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“Contrôle des Ondes”

Voie “Analyse Théorique”

A. ROZANOVA-PIERRAT

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Introduction

It is the coursebook of the 2nd year CentraleSupélec courses “Contrôle des ondes, voie Analyse Théorique”.

We are focused on the engineering problem: a conception of the most efficient anti-noise absorbing barrier, especially for the low frequencies. It is known that in acoustics, an absorbing material is a porous material. As any realistic problem, it is a complex problem, mixing the construction constraints, the physics (of the wave propagation in the air and the porous materials), the choice of absorbent material and its properties, the modeling of the problem, the choice of the model adapted for a fixed application. Then we consider the mathematical analysis of this model and its properties, make the choice of a method allowing the optimization, and in the end, perform the visualization, at least numerically, of an optimal shape of the barrier, and, ideally, the comparison with the experimental measurements.

The complexity of the problem is also related to a vast number of theoretical questions, which are partially shown in this coursebook, which are still entirely or partially open.

Taking the main ideas coming from physics and numerics, we explain how the mathematical methods can treat this shape optimization problem and how some engineer concepts could be explained or improved using a mathematical approach.

We make attention that the course does not pretend to give the best solution to this problem but explain some mathematical methods on the most simple models and show that modern mathematics can help develop and solve engineer problems on this particular example. At the same time, we will also see the limits of the known up of today’s mathematical results, posing numerous open problems.

Therefore, it is crucial to understand “what happens” useful in modern mathematics to solve this engineering problem, understand the main physical ideas, and then apply them to conceive an optimal shape of an absorbent acoustical or electromagnetic wall.

Almost all modern mathematics methods, especially models described by a partial differential equation, are based on Functional Analysis. For instance, some notions of it, the Lebesgue and Sobolev spaces, have been seen in the 1st year. The course subject is presented in 7 Chapters, and some formulated theorems are proven on TDs.

These 7 Chapters and TDs with a numerical project on the eigenmodes localization are sufficient and necessary for the final exam and the “EI” one-week project.

In Chapter 1, we recall the main facts from the 1st year course “EDPs” and show the passage from the one-dimensional case to the multidimensional case with the importance of the boundary regularity on PDE’s solution.

In Chapter 2, we show from one hand the importance of fractal geometry as an efficient model and, on the other hand, how to solve the PDEs on domains with fractal boundaries.

In Chapter 3, we present the main wave propagation models with energy dissipation and the models of the wave propagation in the porous medium. Different experimental parameters of the porous media are also given.

In Chapter 4, we introduce the localization phenomena of eigenmodes for domains with “irregular” boundaries and relate them with the energy dissipation.

In Chapter 5, we give a small survey of what it is the controllability of the PDEs and then of wave propagation problems. In Chapter 5, we also relate this course, on the optimal control of the waves, with the commune course “Automatique et contrôle”. This chapter can be viewed as the last introductory chapter.

Chapter 6 is deduced to introduce all needed new notions of Functional Analysis, not seen in the 1st year, which we use not only for the well-posedness questions, but also to be able to start the parametric (in Chapter 7) shape optimization for a fixed frequency and on a fixed frequency range.

In Chapter 7, the parametric optimization is adapted, for instance, for the optimization of perforated panels, and the geometrical shape optimization is adapted to find an optimal shape of an absorbing wall. We show the existence of optimal distributions theoretically. The theory is illustrated by several numerical results of given numerical algorithms. For an example of a partially absorbing wall filled with 50% of absorbing material, we show (numerically) that there exists a distribution of porous material that absorbs more efficiently ($\approx 31\%$ better) the acoustical energy than the totally absorbent wall (filled with 100% of absorbing material, see Subsection 7.7.2).

As the practical problem is to have an efficient anti-noise wall in a range of frequencies, we investigate this question physically and numerically. We also give an example of the most absorbing wall for the highway, called ‘Fractal Wall’ TM.

Attention: Please be aware that it is impossible to give **all** mathematical proofs during the course. However, if it is not presented as an open problem, all proof can be found in the literature. If you are interested, you are welcome to explore the given 69 references and to go to others. I will be pleased to discuss them with you outside the contributed time for the course. Please send me an email. The attributed hours for this course will mainly allow for presenting the essential ideas and not too complicated proofs, formulating methods, and discussing their applications. All the most technical proofs will be done during TDs. For a more detailed functional analysis presentation, you can read my other course book, “Functional Analysis,” available on Edunao. All these additional studies are not required by the program of the exam.

0.1 Main physical highlights for the optimal control of waves

1. If we change the shape of an absorbing wall, it will modify the wave absorption.

2. To have an optimal interaction between a wave of the wavelength λ and the absorbing wall, the size of the wall's fragments must be equal to $\lambda/2$.
3. To be efficient at different frequencies, the absorbing wall must have multiscaling structures (with sizes related to the half wavelengths).
4. Fractal or pre fractal shapes give examples of a model of multiscaling structures.
5. High-frequency wave propagation can be modeled by the laws of the geometrical optic. Thus, structures, which could capture the rays or make significant the number of the wave interactions with the absorbing wall, are efficient in the high-frequency energy absorption.
6. Porous media are natural media of high-frequency acoustical wave absorption when its wavelength becomes comparable to the sizes of the holes/inhomogeneities in the porous media.
7. For low frequencies, the wavelengths $\lambda = \frac{2\pi}{\omega}$ can become too important, which makes unrealistic the construction of an absorbing wall with fragments of the size $\lambda/2$. Generally, the low frequencies are the most difficult for the efficient absorption.

0.2 Notations

X is a set or a space.

$\emptyset$ is the empty set.

$\overline{X}^G$ is the closure of a set X in the space G (if G is a vector normed space, the closure is by a norm of G).

$\overline{M}$ if M a subset or subspace of X , $\overline{M}$ is the closure of M in the topology of X (fixing a convergence in X , we add all limits to M of sequences of elements from M).

$\overline{a}$ if a is a complex number or a complex valued matrix, then $\overline{a}$ is its complex conjugated, *i.e.* for $a = 1 + 2i$ we have $\overline{a} = 1 - 2i$.

$A \subset B$ means that a set A is a subset of a set B and A can be equal to B .

$A \subsetneq B$ means a set A is a proper subset of B ($A \neq B$).

$\text{Im } A$ is the image of an operator A .

$\text{Ker } A$ is the kernel of an operator A .

$\Rightarrow$ means the uniform convergence.

$\rightarrow$ means the strong convergence.

$\rightharpoonup$ means the weak convergence.

$\overset{*}{\rightharpoonup}$ means the weak* convergence.

$\text{supp } f$ means the support of a function f , *i.e.* the closure of the set $\{x \in \mathbb{R}^n \mid f(x) = 0\}$.

(X, d_X) is a metric space: a set X equipped with a distance d_X .

$\|\cdot\|_X$ is a norm on a vector space X .

$(X, \|\cdot\|_X)$ is a vector normed space: a linear vector space X equipped with a norm $\|\cdot\|_X$.

$d_X(x, y)$ denotes a distance between two elements x and y in the metric space (X, d_X) . An example of a distance in the normed vector space $\|\cdot\|_X$ is $d_X(x, y) = \|x - y\|_X$.

$B_r(a)$ is an open ball of radius r centered in the point a . If $(X, \|\cdot\|)$ is a normed space and $B_r(a) \subset X$, then

$$B_r(a) = \{x \in X \mid \|x - a\|_X < r\}.$$

$\overline{B_r(a)}$ is a closed ball of radius r centered in the point a of a vector normed space $(X, \|\cdot\|)$:

$$\overline{B_r(a)} = \{x \in X \mid \|x - a\|_X \leq r\}.$$

H is usually used to denote a Hilbert space.

$\mathcal{L}(X, Y)$ is the space of all linear continuous operators from X to Y .

X^* is the dual space to X : $X^* = \mathcal{L}(X, \mathbb{R})$.

$\langle f, g \rangle_{X^*, X}$ is $f(g)$ (the value of the functional $f \in X^*$ on the element $g \in X$).

$(f, g)_H$ is the inner product in H .

$\text{Span}(e)$ is the set of finite linear combinations of elements of e .

I is the identity operator $I(x) = x$.

$\mathcal{D}'(\Omega)$ is the dual space to the space $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, named the space of distributions.

D^α is the derivative in the sense of distribution ($\alpha \in \mathbb{N}^n$).

$\nabla u \cdot \nabla v$ by $\cdot$ it is denoted the inner product of two vectors of $\mathbb{R}^n$: $\nabla u \cdot \nabla v = \sum_{i=1}^n \partial_{x_i} u \partial_{x_i} v$.

Ω is a domain (open connected set) of $\mathbb{R}^n$.

$\partial\Omega$ is the boundary of Ω .

μ is a measure on the boundary $\partial\Omega$.

ν is the external normal to the boundary.

Γ_{Dir} is a part of the boundary $\partial\Omega$ on which the Dirichlet boundary condition is imposed.

Γ_{Neum} is a part of the boundary $\partial\Omega$ on which the Neumann boundary condition is imposed.

Γ_{Rob} or simply Γ is a part of the boundary $\partial\Omega$ on which the Robin boundary condition is imposed (in the geometrical shape optimization it is a moving part of $\partial\Omega$).

Chapter 1

Introduction to Partial Differential Equations (PDE)

1.1 Domain and its boundary

We consider a domain $\Omega \subset \mathbb{R}^n$ for $n = 1, 2, 3$ or even it is possible formally to take $n \in \mathbb{N}^*$. Here $\mathbb{N}$ is the set of all natural numbers. $\mathbb{N}^*$ means that 0 is not included.

When we say that Ω is a domain this means:

Definition 1.1.1. A subset Ω of $\mathbb{R}^n$ is called **a domain** if Ω is open and connected (any two points of Ω can be joined by a path, actually, by a continuous line belonging to Ω).¹ See Fig. 1.1.

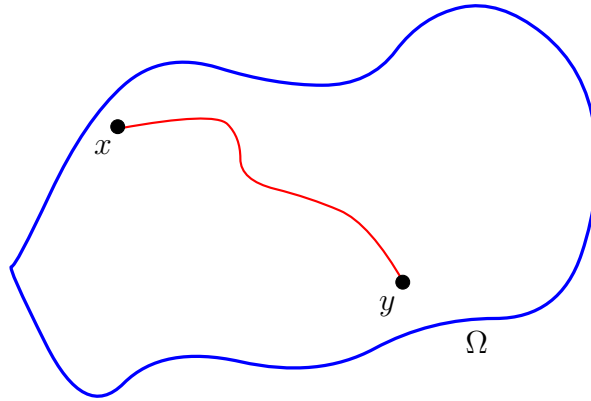


Figure 1.1 – Example of a connected set Ω : all points x and y in Ω can be joined by a continuous line (in red) belonging to Ω .

We notice that D is an open set in $\mathbb{R}^n$ if for all point x belonging to D there exists a radius $r > 0$ of an open ball $B_r(x)$ centered at x such that

$$B_r(x) \subset D.$$

By an open ball in $\mathbb{R}^n$ it is understood

$$B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\|_{\mathbb{R}^n} < r\}.$$

¹ $\mathbb{R}^n$ endowed with its natural topology is an example of a topological space where there is no difference between the notion “connected” and “connected by arcs”. It is used in definition 1.1.1.

If there is not a special precision, by the norm $\|x\|_{\mathbb{R}^n}$ is understood the Euclidean norm

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n \quad \|x\|_{\mathbb{R}^n} = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}.$$

A set $S \subset \mathbb{R}^n$ is called **closed** if its complement $\mathbb{R}^n \setminus S$ is an open set.

In what follows, we always suppose that Ω is a domain in $\mathbb{R}^n$.

If Ω is bounded (**bounded** in the sense that there exists a non trivial ball $B_r(x)$ in $\mathbb{R}^n$ such that $\Omega \subset B_r(x)$, i.e. of a finite n -dimensional volume), then it has a boundary:

Definition 1.1.2. *The intersection of the closure of Ω with the closure of its complement $\mathbb{R}^n \setminus \Omega$ is called **boundary** of Ω and denoted by $\partial\Omega = \overline{\Omega} \cap \overline{\mathbb{R}^n \setminus \Omega}$. A point $x \in \partial\Omega$ is called **boundary point** of Ω .*

Example 1.1.1. • In $\mathbb{R}$, the domain $]0, 1[$ has two boundaries points 0 and 1. The set $\{0\} \cup \{1\}$ is the boundary of $]0, 1[$.

• In $\mathbb{R}^2$, let $\Omega = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$. Then $\partial\Omega = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

Remark: If U is an open set of $\mathbb{R}^n$, then $\overline{U}$ is a closed set of $\mathbb{R}^n$.

In section 1.5 we will see that the form or the regularity of the boundary it is important for the properties of solutions of the PDEs.

1.2 Differential operators

1.2.1 Derivation in the classical sense

Considering $x \in \mathbb{R}^n$ we write in different notations

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} = \partial_{x_1}^2 + \dots + \partial_{x_n}^2 = \operatorname{div} \nabla,$$

the Laplacian operator, which if we apply to a scalar valued function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ which in addition $u \in C^2(\mathbb{R}^n)$ (two times continuously differentiable over all variables), then $\Delta u \in C(\mathbb{R}^n)$ and the derivation is done in the usual classical sense.

Here ∇ and div are first order derivative operators: for a scalar valued function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ ∇u is the vector function $\mathbb{R}^n \rightarrow \mathbb{R}^n$:

$$\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \dots, \partial_{x_n} u)^t.$$

The divergence operator div take a vector of functions $(v_1, \dots, v_n)^t$ and associates a scalar function $\partial_{x_1} v_1 + \partial_{x_2} v_2 + \dots + \partial_{x_n} v_n$. Thus, it is easy to see, that for a scalar function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ (or to the complex numbers $\mathbb{C}$) $\operatorname{div} \nabla u = \Delta u$.

To be able to derive a function, it must have the necessary properties to be derivable. These properties are usually described by an assumption on the function to belong (or not) to a functional

space. It means we define a set of functions having all the same property: as previously, $C^2(\mathbb{R}^n)$ is the set of all functions from $\mathbb{R}^n$ to $\mathbb{R}$ (or $\mathbb{C}$) which are two times continuously differentiable. This set is a normed vector space with the norm

$$\|u\|_{C^2(\mathbb{R}^n)} = \max_{i=1,2,3} \sup_{x \in \mathbb{R}^n} |\partial_{x_j}^{i-1} u(x)|.$$

If we have a bounded domain Ω , then to have the first derivatives of a function u in a classical sense we need to ensure that $u \in C^1(\overline{\Omega})$, i.e u must be a differentiable function up to the boundary.

Remarks on the spaces of continuous functions on compacts or not

Attention: If Ω is a bounded domain, it means that $\overline{\Omega}$ is a compact of $\mathbb{R}^n$. As $\mathbb{R}^n$ is a finite-dimensional space (of the dimension n), then, thanks to the Heine-Borel theorem, a set of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. In addition, the continuity of a function u on a compact ensures that

1. it is bounded on it (there exists its minimal and maximal values),
2. it is **uniformly continuous**: if $u : \overline{\Omega} \rightarrow \mathbb{R}$

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x_1, x_2 \in \overline{\Omega} \text{ such that } \|x_1 - x_2\|_{\mathbb{R}^n} < \delta, \quad |u(x_1) - u(x_2)| < \epsilon.$$

Thus, for bounded domains, the functions of the space $C(\overline{\Omega})$ are all bounded and uniformly continuous functions on the compact $\overline{\Omega}$.

If we consider a function f from $C(\Omega)$, this time Ω is an open set, then we only know that f is continuous on Ω . It can be unbounded, and it cannot be uniformly continuous on Ω .

Example 1.2.1. For instance, it is easy to see that for $\Omega =]0, 1[$, $n = 1$, the functions

$$\begin{aligned} f_1(x) = \frac{1}{x} : & \text{ is not bounded and not uniformly continuous } \Rightarrow f_1 \in C(\Omega), \quad f_1 \notin C(\overline{\Omega}) \\ f_2(x) = \sin \frac{1}{x} : & \text{ is bounded but not uniformly continuous } \Rightarrow f_2 \in C(\Omega), \quad f_2 \notin C(\overline{\Omega}). \end{aligned}$$

We have

Proposition 1.2.1. Let K be a compact of $\mathbb{R}^n$. The space $C(K)$ of all continuous functions on K endowed with the norm

$$\|f\|_{\infty} = \sup_{x \in K} |f(x)| \quad \forall f \in C(K),$$

is the Banach space (a complete vector normed space): any Cauchy sequence $(f_j)_{j \in \mathbb{N}}$ converges by this norm, more precisely, uniformly converges, to a element of $C(K)$.

We use $\Rightarrow$ for the uniform convergence of functions.

1.2.2 Derivation in the sense of distributions

There are two ways to define a derivative of a function. If we need to derive it once on x_j and it is continuously differentiable on Ω , then there exists its derivation on x_j in the **classical or strong** sense. Nevertheless, if it is less regular, for instance, from $L^2(\Omega)$ (see the first-year course CIP), it is also possible to define its derivative, but this time in a **weak** sense.

It is known that it is possible to derive infinitely many times any element of $L^2(\Omega)$ in the sense of distributions (see the first-year course EDP). However, for the solving of the PDEs, it is crucial to work in a Hilbert space. For example, we need to work in Hilbert space to apply the Riesz representation theorem or the Lax-Milgram Lemma. Thus, we also need to impose some regularity on the derivatives, even if it is $L^2(\Omega)$ regularity, to have, if possible, a Hilbert space. This goal restricts the space of infinitely differentiable distributions $\mathcal{D}'(\Omega)$ to much smaller spaces as $H^1(\Omega)$ or to its subsets $H_0^1(\Omega)$, $H^2(\Omega)$ and others.

Remark: The spaces $L^p(\Omega, \mu)$ were introduced in the first year course CIP: if μ is a positive measure, σ -additive and $\Omega \subset \mathbb{R}^n$, then

$$L^p(\Omega, \mu) = \{ \mu - \text{measurables classes of functions } f : \Omega \rightarrow \mathbb{C} \mid \|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p d\mu \right)^{\frac{1}{p}} \},$$

where it is supposed that all two functions equal μ -almost everywhere belong to the same class and thus associate to the same element of $L^p(\Omega, \mu)$. In what follows, we will use these spaces referring to CIP course and use them not only on the whole domain Ω but also on its boundary $\partial\Omega$, on which we will specify the measure μ as a d -dimensional measure (see Chapter 2). The advantage of working in $L^2(\Omega, \mu)$ is that it is a Hilbert space endowed with the inner product:

$$\langle u, v \rangle_{L^2(\Omega, \mu)} = \int_{\Omega} u \bar{v} d\mu, \quad \forall u, v \in L^2(\Omega, \mu),$$

where by $\bar{v}$ is denoted the complex conjugate of v .

Space $H^1(\Omega)$, or equivalently $H^1(\Omega)$, is the Sobolev space introduced in the first year course EDPs:

$$H^1(\Omega) = \{ u \in L^2(\Omega) \mid \text{the distribution } \nabla u \text{ belongs to } [L^2(\Omega)]^n \}.$$

We recall that $u \in L^2(\Omega) \subset L_{loc}^1(\Omega)$ (and thus it is a regular distribution) has in the sense of distributions the first order derivatives $\nabla u \in [L^2(\Omega)]^n$ (here we impose the regularity L^2 for all distributions defining the first derivatives of u), if

$$\forall \phi \in \mathcal{D}(\Omega) \quad \int_{\Omega} \partial_{x_i} u \phi dx = - \int_{\Omega} u \partial_{x_i} \phi, \quad i = 1, \dots, n.$$

Remark: If the derivative exists in the classical sense, then the derivative in the sense of distributions is equal to the classical one.

For more details on the distributions, see the first-year course EDP.

Attention: For any domain Ω (bounded or not) we obviously have

$$H^1(\Omega) \subset L^2(\Omega) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega).$$

with all strict inclusions.

The usual norm on $H^1(\Omega)$ is defined by

$$\|v\|_{H^1(\Omega)} = \left(\int_{\Omega} |v|^2 dx + \int_{\Omega} |\nabla v|^2 \right)^{\frac{1}{2}} = (\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2)^{\frac{1}{2}} \quad (1.1)$$

and it is associated with the inner product

$$\langle u, v \rangle_{H^1(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}.$$

With this inner product, $H^1(\Omega)$ is a Hilbert space.

Attention: If $\Omega \subset \mathbb{R}$, *i.e.* in the one-dimensional case $n = 1$, it is known that $H^1(\Omega) \subset C(\overline{\Omega})$. But it is not true any more for $n \geq 2$!

However, for all domains Ω of $\mathbb{R}^n$, $n \in \mathbb{N}^*$ the space $C^\infty(\Omega)$ is dense in $H^1(\Omega)$.

1.3 Integration by parts or Green's formulas

1.3.1 One dimensional case (1D)

If f and g are functions defined on $]a, b[\subset \mathbb{R}$ (for a simplicity we suppose that $]a, b[$ is bounded), the integration by parts formulas

$$\int_{]a, b[} f' g dx = f g \Big|_{x=a}^{x=b} - \int_{]a, b[} f g' dx \quad (1.2)$$

holds if f and g are $C^1([a, b])$ functions (continuously differentiable and continuous on the compact $[a, b]$). Thus the Lebesgue integrals are equal to the Riemann integrals (see course CIP of the first year).

For the well posedness of Lebesgue integrals, it is sufficient to impose f' and g from $L^2(]a, b[)$ to ensure the well posedness of $\int_{]a, b[} f' g dx$ (the well posedness here means that the integral exists, *i.e.* it is finite). In the same way, we need to have f and g' in $L^2(]a, b[)$ to ensure the well posedness of $\int_{]a, b[} f g' dx$. Indeed, by the Cauchy-Schwarz inequality, as soon as f and g' in $L^2(]a, b[)$,

$$\int_{]a, b[} f g' dx \leq \|f\|_{L^2(]a, b[)} \|g'\|_{L^2(]a, b[)} < \infty.$$

More briefly, we need to have f and g from $H^1(]a, b[)$. But we also need to know the values on the boundary of f and g , since in (1.2) we need to calculate $f(b)g(b) - f(a)g(a)$. As in the one

dimensional case all elements of $H^1(]a, b[)$ are continuous up to the boundary ($H^1(]a, b[) \subset C([a, b])$), then $f(b)g(b) - f(a)g(a)$ is well defined as involves the finite values of a bounded and continuous on the compact $[a, b]$ functions.

Remark: Knowing the function f (and g) only on $]a, b[$, it is possible to define the values in points a and b by the continuity. These values of f are called **a trace** of the function f on the boundary $\{a\} \cup \{b\}$.

1.3.2 2D and general case

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a compact boundary $\partial\Omega$. This time $\partial\Omega$ is a curve of $\mathbb{R}^2$ (or more generally a subset of $\mathbb{R}^2$ whose 2-dimensional Lebesgue measure is equal to 0). Considering u and v on Ω , there is a question how to define them on $\partial\Omega$. If u and v are in $C(\overline{\Omega})$ the question can be solved as in the one dimensional case. The main problem is when we consider elements of $H^1(\Omega)$ in $\mathbb{R}^n$ with $n \geq 2$. As it was mentioned previously, it is the case when the embedding $H^1(\Omega) \subset C(\overline{\Omega})$ does not hold.

In this situation, we need to generalize the notion of the trace of a continuous function to a trace of an element of $L^1_{loc}(\Omega)$. This means to define the trace for regular distributions.

In the classical sense

Let us start by the classical case when u and v belong to $C^2(\overline{\Omega})$. In particular, we also suppose that the boundary can be locally described by a non-trivial twice continuously differentiable function. Here “locally” means that for all boundary points $x \in \partial\Omega$ there exists a non trivial ball B_r containing x , such that $\partial\Omega \cap B_r$ is a graph of a C^2 function:

Definition 1.3.1. The boundary $\partial\Omega$ is called **a regular boundary of the class C^k** ($k \geq 1$, see Fig. 1.2), if for all $x_0 \in \partial\Omega$ there exist $r > 0$ and $f(x) \in C^k(B_r(x_0))$ such that

$$\{x \in B_r(x_0) \mid f(x) = 0\} = \partial\Omega \cap \overline{B_r(x_0)}, \quad df|_{B_r(x_0)} \neq 0.$$

By $df|_{B_r(x_0)}$ is denoted the differential of f considered only on $B_r(x_0)$ (i.e. only for $x \in B_r(x_0)$).

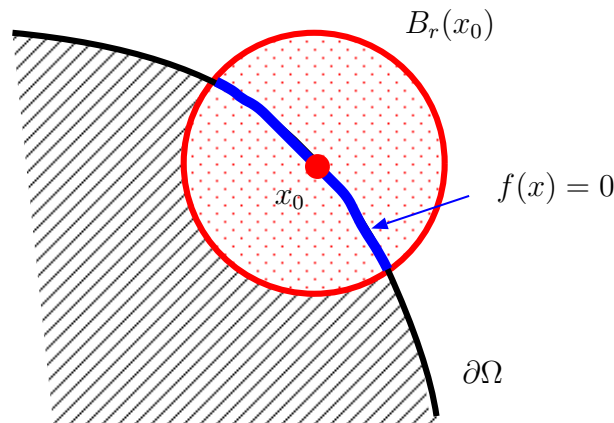


Figure 1.2 – Example of a regular boundary from Definition 1.3.1.

In addition we define the (locally) Lipschitz boundaries (see also [41, Definition 2.4.5]) with a geometrical parametrization):

Definition 1.3.2. Let Ω be a bounded domain of $\mathbb{R}^n$. Its boundary $\partial\Omega$ is **(locally) Lipschitz** if

$$\forall x \in \partial\Omega \quad \exists \text{ neighborhood } U_x : \partial\Omega \cap U_x \text{ is the graph of a Lipschitz continuous function } f,$$

and, Ω lies on one side of its boundary.

Remark: Here by a Lipschitz continuous function we understand $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ which is continuous and there exists $M > 0$ such that

$$\forall x, y \in \mathbb{R}^{n-1} \quad |f(x) - f(y)| \leq M \|x - y\|_{\mathbb{R}^{n-1}}.$$

Example 1.3.1. A typical example of a Lipschitz boundary is a polygon domain, i.e. a piecewise regular boundary with angle points. The other example is a prefractal domain constructed with a finite number of fractal generations, as shown in Fig. 1.3. In the angle points, the first derivative does not exist in the classical sense. It is the case when the first derivative (and also the normal unit vector) exists almost everywhere: for all $x \in \partial\Omega$ except for a finite number of points (or more generally of a zero measure set of points).

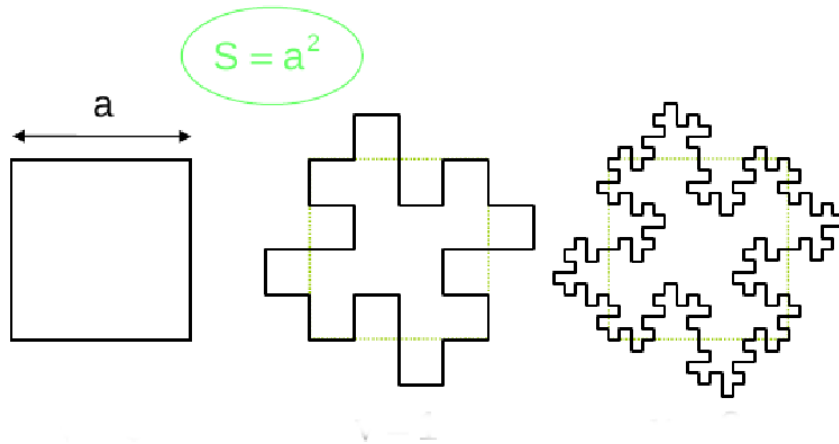


Figure 1.3 – Example of Lipschitz boundaries for a prefractal domain for a shape which, if the prefractal iterations are repeated infinitely many times, is called fractal of Minkowski or the square Koch curve. These shapes' main property is that they are symmetrical and thus keep the initial volume a^2 constant for each new prefractal iteration. The figure is by B. Sapoval.

For C^1 boundary $\partial\Omega$ it is possible to define the unit normal vector and the normal derivative in the following *strong* sense:

Definition 1.3.3. If $\partial\Omega$ is C^1 , then for any point $x_0 \in \partial\Omega$ is defined the outward pointing **unit normal** vector field $\nu(x_0) = (\nu_1, \dots, \nu_n)$, the vector of the norm $\|\nu(x_0)\|_{\mathbb{R}^n} = 1$ perpendicular to the tangent plane to $\partial\Omega$ at the point x_0 and oriented outside of Ω (see Fig. 1.4).

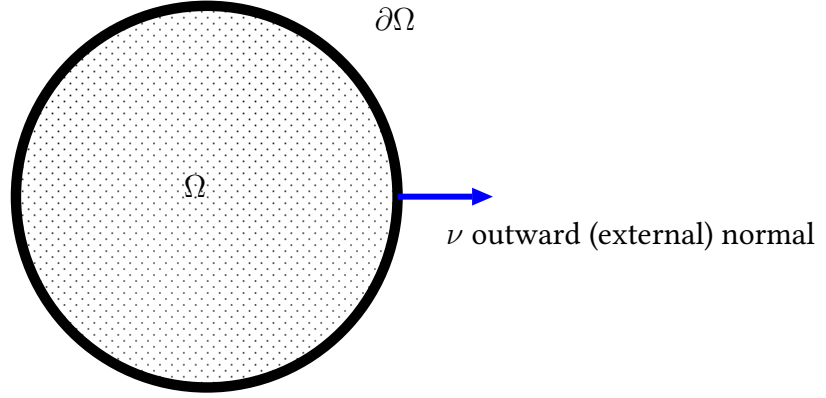


Figure 1.4 – A ball gives an example of domain with a C^∞ boundary on which exists the outward unit normal for all $x \in \partial\Omega$.

Definition 1.3.4. If $u \in C^1(\overline{\Omega})$, then we call $\frac{\partial u}{\partial \nu}(x_0) := \nu(x_0) \cdot \nabla u$ for $x_0 \in \partial\Omega$ the (outward) normal derivative of u on $\partial\Omega$.

We notice that the differential operator Δ can be defined in the classical sense from $C^2(\overline{\Omega})$ to $C(\overline{\Omega})$.

Thus there holds the following Green's formulas or integration by parts:

Theorem 1.3.1. Let $u \in C^2(\overline{\Omega})$, $v \in C^1(\overline{\Omega})$ and μ be the boundary measure on $\partial\Omega$. Then

$$\int_{\Omega} \Delta u v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\mu - \int_{\Omega} \nabla u \cdot \nabla v dx. \quad (1.3)$$

For $u, v \in C^1(\overline{\Omega})$ it holds the usual integration-by-parts formula:

$$\int_{\Omega} (\partial_{x_i} u) v dx = \int_{\partial\Omega} u v \nu_i d\mu - \int_{\Omega} u (\partial_{x_i} v) dx \quad (i = 1, \dots, n). \quad (1.4)$$

In the weak sense

If $\partial\Omega$ is Lipschitz, then the normal vector exists almost everywhere on the boundary. If $\partial\Omega$ is a self-similar fractal, *i.e.* the limit boundary obtained by infinitely many pre-fractal Lipschitz generations, then there does not exist a boundary point in which it is possible to construct the normal vector by Definition 1.3.3. All points in a fractal $\partial\Omega$ are irregular in the classical sense: the classical derivative does not exist. Thus, we need to use the notion of the derivative in the sense of distribution. It is also possible to define the normal derivative on an irregular boundary, but this time in the weak or distributional sense. For more information about fractal boundaries, see Chapter 2. Here we don't care about the form or kind of fractal boundary and how it could be constructed, but we only use the fact that it has a known dimension d ($n-1 \leq d < n$, $d \in \mathbb{R}$).

Remark:

- If $\partial\Omega \in C^\infty$ or in $\mathbb{C}^k$ ($k \in \mathbb{N}$) or is Lipschitz, then its dimension is equal to $n - 1$. These are examples of $(n - 1)$ -sets. The typical measure on them is the $(n - 1)$ -dimensional Lebesgue measure.
- If $\partial\Omega$ has a bigger dimension, equal to a real number d ($n - 1 < d < n$), then $\partial\Omega$ is a non-Lipschitz (possibly fractal) boundary, given by a d -set. On a d -set boundary with $d > n - 1$ the $(n - 1)$ -dimensional Lebesgue measure is not finite: the length of a fractal curve in 2D is infinite. But any d -dimensional measure on a d -set is finite. An example of a d -measure is the d -dimensional Hausdorff measure m_d (its definition is not considered in this course, but it is useful just to mention its existence) [48].

This is a rigorous definition of a d -set:

Definition 1.3.5 (Ahlfors d -regular set or d -set [49, 50, 69, 68]). *Let F be a Borel non-empty subset of $\mathbb{R}^n$. The set F is called a d -set, or a d -dimensional set, ($0 < d \leq n$) if there exists a d -measure μ on F , i.e. a positive Borel measure (defined for all Borel sets) with support F ($\text{supp } \mu = F$, i.e. for all $A \subset \mathbb{R}^n$ $A \cap F = \emptyset \Rightarrow \mu(A) = 0$) such that there exist constants $c_1, c_2 > 0$,*

$$c_1 r^d \leq \mu(F \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall x \in F, 0 < r \leq 1,$$

where $\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|x - y\|_{\mathbb{R}^n} \leq r\}$ denotes the closed Euclidean ball centered at x and of radius r .

Remark: By [49, Prop. 1, p 30] all d -measures on a fixed d -set F are equivalent. It is also possible to define a d -set by the d -dimensional Hausdorff measure m_d (we don't consider the definition of the Hausdorff measure in this course). In particular, F has the Hausdorff dimension d in the neighborhood of each point of F [49, p.33].

More precisely, let Ω be a bounded domain, and $\partial\Omega$ be its Ahlfors d -regular boundary (for more general boundaries see [44, 45]).

Let in addition μ be a positive finite (in the sense that $\mu(\partial\Omega) < \infty$) d -measure on the boundary $\partial\Omega$ (if $\partial\Omega$ is Lipschitz, then μ is the usual $(n - 1)$ -dimensional Lebesgue measure). Then it is possible to define as in Remark 1.2.2

$$L^2(\partial\Omega, \mu) = \{ \mu - \text{measurables classes of functions } f : \partial\Omega \rightarrow \mathbb{C} \mid \|f\|_{L^2(\partial\Omega, \mu)} = \left(\int_{\partial\Omega} |f(x)|^2 d\mu \right)^{\frac{1}{2}} < \infty \}, \quad (1.5)$$

where, as usually, it is supposed that all two functions equal μ -almost everywhere belong to the same class and thus associate to the same element of $L^2(\partial\Omega, \mu)$.

A particular example of a d -set boundary is a self-similar fractal of the dimension d (not all d -sets are self-similar). In what follows, we always suppose that we work with bounded domains $\Omega \subset \mathbb{R}^n$ with a self-similar fractal d -set boundary ($d \in]n - 1, n[$) or Lipschitz boundary if $d = n - 1$.

Remark: This choice is motivated by the fact [17, 66, 69] that these kind of domains are examples of H^1 -extension domains [37]: i.e. there exists a bounded linear extension operator $E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$:

$$(i) \quad \forall u \in H^1(\Omega), \quad Eu|_{\Omega} = u,$$

$$(ii) \quad \forall u \in H^1(\Omega), \quad \|Eu\|_{H^1(\mathbb{R}^n)} \leq C\|u\|_{H^1(\Omega)},$$

where $C > 0$ depends only on Ω .

Initially, it was known that the Lipschitz domains are H^1 -extension domains [17, 66]. Then, it was generalized by [47] for $n = 2$ and by [37] for all n for a more general class of domains containing not only Lipschitz domains, but also domains with fractal boundaries.

Thanks to [37], it is known that all H^1 -extension domains are n -sets (a d -set with $d = n$):

$$\exists c > 0 \quad \forall x \in \overline{\Omega}, \quad \forall r \in]0, \delta[\cap]0, 1[\quad \lambda^n(B_r(x) \cap \Omega) \geq C\lambda^n(B_r(x)) = cr^n,$$

where $\lambda^n(A)$ denotes the (n -dimensional) Lebesgue measure of a set A in $\mathbb{R}^n$. This property is also called the measure density condition [37]. Let us notice that an n -set Ω cannot be “thin” close to its boundary $\partial\Omega$, since it must all times contain a non trivial ball in its neighborhood.

If we have an element of $u \in H^1(\Omega)$ it is possible to define its **trace** on $\partial\Omega$ locally (point by point) μ -almost everywhere (a.e.) in the following way (a definition-theorem by Wallin (1991) [69] and also by a more recent work [22]):

Definition 1.3.6. For a domain Ω of $\mathbb{R}^n$ with a self-similar or Lipschitz boundary $\partial\Omega$ defined as the support of a d -measure μ , the trace operator $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega, \mu)$ is defined μ -a.e. by

$$\text{Tr}u(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) dy, \quad x \in \partial\Omega.$$

Here, as previously, λ^n is the n -dimensional Lebesgue measure.

In the last definition we take a function u defined for $x \in \Omega$ and define its values, i.e. its trace on the boundary $\partial\Omega$ by formula (1.3.6).

Therefore, it is possible to consider the trace of elements of $H^1(\Omega)$ without taking into account if these elements are continuous or not.

Definition 1.3.7. The space $B(\partial\Omega) := \text{Tr}(H^1(\Omega))$ is the image of the trace operator considered from $H^1(\Omega)$.

In what follows we use without proof the following particular case of the trace theorem due to Jonsson and Wallin in the years of 1990:

Theorem 1.3.2. Let Ω be a bounded domain in $\mathbb{R}^n$ with a self-similar or Lipschitz boundary $\partial\Omega$ defined as the support of a d -measure μ , $n - 1 \leq d < n$. Then

1. The space $B(\partial\Omega)$ is the proper (dense) Hilbert subspace of $L^2(\partial\Omega, \mu)$, endowed with the norm

$$\|\varphi\|_{B(\partial\Omega)} := \min\{\|g\|_{H^1(\Omega)} \mid \varphi = \text{Tr } g\}. \quad (1.6)$$

2. $\text{Ker } \text{Tr} = H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{H^1}$ is the closure in H^1 of the set of indefinitely differentiable functions with compact support in Ω .

3. The trace operator $\text{Tr} : H^1(\Omega) \rightarrow B(\partial\Omega)$ is linear continuous and, in particular, there exists a constant $C > 0$ (depending only on n and d) such that for all $u \in H^1(\Omega)$

$$\|\text{Tr}u\|_{L^2(\partial\Omega, \mu)} \leq C\|u\|_{H^1(\Omega)}. \quad (1.7)$$

4. There exists a continuous linear extension operator $E : B(\partial\Omega) \rightarrow H^1(\mathbb{R}^n)$ such that $\text{Tr}(Ef) = f$, $f \in B(\partial\Omega)$ (the right inverse of the trace operator).

Remark:

1. Since, $B(\partial\Omega) \subset L^2(\partial\Omega, \mu)$ then for some constant $C > 0$

$$\|\text{Tr}u\|_{L^2(\partial\Omega, \mu)} \leq C\|\text{Tr}u\|_{B(\partial\Omega)}.$$

2. The continuity of the trace operator $\text{Tr} : H^1(\Omega) \rightarrow B(\partial\Omega)$ reads (by using (1.6)):

$$\|\text{Tr}u\|_{B(\partial\Omega)} \leq \|u\|_{H^1(\Omega)} \quad (1.8)$$

Combining these two points we have estimate (1.7).

More generally, we take in mind that

Remark:

1. If A is a linear continuous operator from a normed vector space X to a vector normed space Y , then its continuity is equivalent to its boundness, *i.e.* there exists a constant $C > 0$ such that for all $x \in X$

$$\|Ax\|_Y \leq C\|x\|_X.$$

The infimum of the possible constants C gives the operator norm of A , denoted by $\|A\|_{\mathcal{L}(X,Y)}$. For more details see TD 1 Problème 6.

2. Any operator considered on its image is surjective.
3. The trace operator on the boundary is not injective:

$$\text{Ker}(\text{Tr}(H^1(\Omega))) = H_0^1(\Omega) = \{u \in H^1(\Omega) \mid \text{Tr}u = 0 \text{ on } \partial\Omega\}.$$

4. The extension operator $E : B(\partial\Omega) \rightarrow H^1(\mathbb{R}^n)$ is also called the right inverse operator because of only one identity $\text{Tr} \circ E = \text{Id}$ (since $\text{Ker}(\text{Tr}(H^1(\Omega))) \neq \{0\}$, the identity $E \circ \text{Tr} = \text{Id}$ does not hold).
5. Let $V_1(\Omega)$ be the orthogonal complement space to $H_0^1(\Omega)$ (the kernel of the trace operator) in $H^1(\Omega)$:

$$H^1(\Omega) = H_0^1(\Omega) \oplus V_1(\Omega).$$

Then, the restriction of the trace operator on $V_1(\Omega)$ $\text{Tr}|_{V_1(\Omega)} : V_1(\Omega) \rightarrow B(\partial\Omega)$ is bijective (and, as previously, linear continuous). The space $V_1(\Omega) = \{u \in H^1(\Omega) \mid -\Delta u + u = 0 \text{ in } \mathcal{D}'(\Omega)\}$ is the space of 1-harmonic functions.

Definition 1.3.8. The set $B'(\partial\Omega) = \mathcal{L}(B(\partial\Omega), \mathbb{C})$ is the dual space to $B(\partial\Omega)$, *i.e.* the set of all linear continuous functionals (or forms) from $B(\partial\Omega) \rightarrow \mathbb{C}$. We denote the value of $F \in B'(\partial\Omega)$

on an element u of $B(\partial\Omega)$ by

$$\langle F, u \rangle_{(B'(\partial\Omega), B(\partial\Omega))} := F(u) \in \mathbb{C}.$$

As F is a linear continuous (or bounded) functional, then there exists a constant $C > 0$ such that for all $u \in B(\partial\Omega)$

$$|\langle F, u \rangle_{(B'(\partial\Omega), B(\partial\Omega))}| \leq C \|u\|_{B(\partial\Omega)}.$$

In addition, there holds

$$B(\partial\Omega) \subset L^2(\partial\Omega, \mu) \subset B'(\partial\Omega). \quad (1.9)$$

Knowing the properties of the trace operator, it is possible to generalize the Green formulas of Theorem 1.3.2 in the following weak sense (initially due to Lancia in the years of 2000).

Proposition 1.3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a d -set compact boundary $\partial\Omega$ on which it is defined a d -measure μ , $n - 1 \leq d < n$. Then*

1. *for all $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ we can define a bounded linear functional $\frac{\partial u}{\partial \nu} \in B'(\partial\Omega)$ by*

$$\langle \frac{\partial u}{\partial \nu}, \text{Tr} v \rangle_{(B'(\partial\Omega), B(\partial\Omega))} := \int_{\Omega} (\Delta u) \bar{v} dx + \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx, \quad (1.10)$$

$$v \in H^1(\Omega).$$

2. *Similarly, for any $u \in H^1(\Omega)$ and $1 \leq i \leq n$, we can define a bounded linear functional $u \cdot \nu_i \in B'(\partial\Omega)$ by*

$$\langle u \cdot \nu_i, \text{Tr} v \rangle_{(B'(\partial\Omega), B(\partial\Omega))} := \int_{\Omega} \frac{\partial u}{\partial x_i} \bar{v} dx + \int_{\Omega} u \frac{\partial \bar{v}}{\partial x_i} dx, \quad (1.11)$$

$$v \in H^1(\Omega).$$

Proof

(it could be omitted)

The statement of proposition follows, thanks to Theorem 1.3.2, from the surjectivity of the linear continuous trace operator $\text{Tr} : H^1(\Omega) \rightarrow B(\partial\Omega)$. Indeed, let us take $v \in H^1(\Omega)$. By Theorem 1.3.2, there exists $\text{Tr} v \in B(\partial\Omega)$ and it holds (1.8).

We define for a fixed $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ the linear functional

$$L : w \in B(\partial\Omega) \mapsto L(w) = \int_{\Omega} Ew \Delta u dx + \int_{\Omega} \nabla(Ew) \cdot \nabla u dx \in \mathbb{C},$$

where by Theorem 1.3.2 the operator $E : B(\partial\Omega) \rightarrow H^1(\mathbb{R}^n)$ is the linear bounded extension operator such that $\text{Tr}(Ew) = w$ μ a.e. Hence, we denote $\mathbb{1}_{\Omega} Ew \in H^1(\Omega)$ by v . Then, by the continuity of the extension operator, we state that L is continuous:

$$\begin{aligned} |L(w)| &\leq \left| \int_{\Omega} v \Delta u dx \right| + \left| \int_{\Omega} \nabla v \cdot \nabla u dx \right| \\ &\leq \left(\|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \right) \|v\|_{H^1(\Omega)} \leq C \left(\|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \right) \|w\|_{B(\partial\Omega)}. \end{aligned}$$

Hence, as $\text{Tr} v = w$, with notations $L(w) = \langle \frac{\partial u}{\partial \nu}, \text{Tr} v \rangle_{(B'(\partial\Omega), B(\partial\Omega))}$ we exactly obtain the generalized Green formula (1.10). $\square$

1.4 Multidimensional boundary valued problems for the Poisson equation

1.4.1 Definitions of different boundary type problems and recall of a definition of a well posed by Hadamard problem

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, to define correctly a PDE problem on it (especially for ensuring its well posedness in the sense of Hadamard), we need to specify a **boundary condition** for $x \in \partial\Omega$.

Considering the example of the Poisson equation $-\Delta u = f$, there is the list of the most common types of boundary conditions:

Definition 1.4.1. Let $\phi(x)$ be a given function. We specially distinct three types of boundary conditions for $u(x)$ with $x \in \partial\Omega$.

1. **Dirichlet boundary conditions** or boundary conditions of the first kind:

$$u|_{\partial\Omega} = \phi(x). \quad (1.12)$$

If $\phi = 0$, condition (1.12) is named the homogeneous Dirichlet condition: $u|_{\partial\Omega} = 0$.

2. **Neumann boundary conditions** or boundary conditions of the second kind:

$$\left. \frac{\partial u}{\partial \nu} \right|_{\partial\Omega} = \phi(x). \quad (1.13)$$

The homogeneous Neumann boundary condition is $\left. \frac{\partial u}{\partial \nu} \right|_{\partial\Omega} = 0$.

3. **Robin boundary conditions** or boundary conditions of the third kind:

$$\left(\frac{\partial u}{\partial \nu} + \alpha u \right) \Big|_{\partial\Omega} = \phi(x), \quad (1.14)$$

where α is a known parameter or a function.

The problem to find u satisfying for $x \in \Omega$ the Poisson equation (or an other PDE) and one of the three boundary conditions above is called the **boundary valued problem**.

Let us also recall what it is a well-posed problem in the sense of Hadamard. Jacques Hadamard named the “good” mathematical models by well-posed problems defined in the following way.

Definition 1.4.2. We say that a given problem for a partial differential equation is **well-posed** (by Hadamard) if

- a) the problem in fact has a solution;
- b) this solution is unique;
- c) the solution depends continuously on the data given in the problem (the solution is **stable**).

If one of these conditions does not hold, the problem is called **ill-posed**.

Obviously not all boundary valued problems are well posed.

The stability is particularly important for problems arising from physical applications: the conditions specifying the problem (initial or/and boundary conditions) are often obtained from the measurements which can never be exact. Therefore, we would prefer that small variations of data in initial or/and boundary conditions correspond to small changes of the unique solution. If a small error in data implies a big difference in solutions, we cannot conclude about a reasonable solvability of the model.

Example 1.4.1. *There is an example of an ill-posed problem of Hadamard for an elliptic equation (the Laplace equation, which is actually the Poisson equation with $f = 0$).*

$$\begin{aligned} \Delta u &= 0 \quad \text{for } (x, y) \in (-\infty, +\infty) \times (0, \delta), \\ u|_{y=0} &= \phi(x), \quad \frac{\partial u}{\partial y}\bigg|_{y=0} = 0 \quad \text{with } \phi(x) = \frac{\cos x}{k}. \end{aligned}$$

The solution of this problem is $u(x, y) = \frac{\cos kx \cosh ky}{k}$. For large k the function ϕ is small, but u is not.

Example 1.4.1 shows that if we add initial conditions to an elliptic equation, it is possible to get an ill-posed problem.

1.4.2 Poisson equation with Dirichlet boundary conditions: from $1D$ to nD

Let Ω be a bounded domain of $\mathbb{R}^n$. Let us consider the Poisson equation

$$-\Delta u = f, \quad \text{on } \Omega \tag{1.15}$$

with for instance the Dirichlet homogeneous boundary conditions:

$$u|_{\partial\Omega} = 0. \tag{1.16}$$

For $n = 1$ the problem becomes

$$-u'' = f, \quad \text{on }]a, b[\subset \mathbb{R} \tag{1.17}$$

$$u(a) = u(b) = 0, \tag{1.18}$$

which has the unique explicit solution. Suppose we chose Ω as a product of intervals, a parallelepiped, or a cylindrical domain or a sphere. In that case, it is possible to express the solution using the method of separation of variables. This time the solution is less explicit since it is given by series. Nevertheless, if the domain Ω is different from these cases, it is no longer possible. However, for the general case of an arbitrary domain Ω of $\mathbb{R}^n$, there is a possibility to prove the existence of a unique weak solution u depending continuously from the initial data (f here).

Indeed, if we take $f \in L^2(\Omega)$, then from (1.15) it follows that $\Delta u \in L^2(\Omega)$. We take $u, v \in H_0^1(\Omega)$ and apply (1.10) to obtain the following variational form:

$$\forall v \in H_0^1(\Omega) \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx. \tag{1.19}$$

We recall that

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^1(\Omega)}} = \{u \in H^1(\Omega) \mid \text{Tr } u = 0 \text{ on } \partial\Omega\}$$

is a proper Hilbert subspace of $H^1(\Omega)$ (it is equal to $H^1(\Omega)$ only if $\Omega = \mathbb{R}^n$, but here it is not allowed as Ω is supposed to be bounded). From the first year course EDP it is known that $H_0^1(\Omega)$ is a Hilbert space endowed with the inner product

$$(u, v)_{H_0^1(\Omega)} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} dx \quad \forall u, v \in H_0^1(\Omega). \quad (1.20)$$

This inner product is associated with the norm $\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)}$, which is equivalent to the usual norm of $H^1(\Omega)$ given in (1.1): there exist constants $C_1 > 0$ and $C_2 > 0$ such that for all $u \in H_0^1(\Omega)$

$$C_1 \|u\|_{H^1(\Omega)} \leq \|u\|_{H_0^1(\Omega)} \leq C_2 \|u\|_{H^1(\Omega)}.$$

Attention: This norm equivalence is due to the Poincaré inequality: for all $u \in H_0^1(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)} \quad (1.21)$$

for a constant $C_P > 0$ depending only on the diameter of Ω . It is important to understand that Poincaré inequality does not hold on $H^1(\Omega)$, which contains all constant functions on Ω . If we take $u = 1$ on Ω then from (1.21) it follows

$$\text{Vol}(\Omega) \leq 0.$$

Since the volume of Ω is strictly positive, we obtain a contradiction.

Working on $H_0^1(\Omega)$, we only have access to the 0 constant, for which the inequality obviously holds. But it is also held if we impose to have zero traces only on a part of the boundary $\partial\Omega$, since, in this case, all constants, except 0, are again excluded.

Thus the variational formulation (1.19) can be written using the inner product of $H_0^1(\Omega)$:

$$\forall v \in H_0^1(\Omega) \quad (u, v)_{H_0^1(\Omega)} = (f, v)_{L^2(\Omega)}. \quad (1.22)$$

The data f is fixed in $L^2(\Omega)$. For all $v \in H_0^1(\Omega)$ the map $v \in H_0^1(\Omega) \mapsto (f, v)_{L^2(\Omega)} \in \mathbb{C}$ is a linear continuous functional on $H_0^1(\Omega)$ (a Hilbert space):

$$|(f, v)_{L^2(\Omega)}| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C_P \|f\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)}. \quad (1.23)$$

Thus we apply the Riesz representation theorem to obtain that there exists unique $u \in H_0^1(\Omega)$ solving (1.19) and in addition it satisfies the a priori estimate (taking $v = u$ in (1.22) and combining with (1.23))

$$\|u\|_{H_0^1(\Omega)}^2 = (f, u)_{L^2(\Omega)} \leq C_P \|f\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)} \quad \Rightarrow \quad \|u\|_{H_0^1(\Omega)} \leq C_P \|f\|_{L^2(\Omega)}.$$

This is the stability estimate ensuring the continuous dependence of the solution u from the data f : if there is a small error in the input data these implies a small error of the solution.

Thus we have proved

Theorem 1.4.1. For an arbitrary bounded domain Ω of $\mathbb{R}^n$ and all $f \in L^2(\Omega)$ there exists unique weak solution $u \in H_0^1(\Omega)$ of the homogeneous Dirichlet problem in the sense of (1.22). In addition, it satisfies

$$\|u\|_{H_0^1(\Omega)} \leq C_P \|f\|_{L^2(\Omega)},$$

where $C_P > 0$ is the constant from the Poincaré inequality.

Remark: Here we don't care at all about the regularity of the boundary of Ω . The only important thing which was used is that Ω is bounded. For the validity of the Poincaré inequality, the domain must be bounded at least in one direction of $\mathbb{R}^n$.

The importance of the boundary regularity comes when we need more regularity of the solution u . Two typical regularities are to be the classical solution, *i.e.* $u \in C^2(\Omega) \cap C(\overline{\Omega})$, or to be simply in $C(\overline{\Omega})$. Let us discuss it in detail in the next section.

1.4.3 Poisson equation with an homogeneous Robin boundary condition

Let us consider the Poisson equation with this time the homogeneous Robin boundary condition on $\partial\Omega$ with a real constant α_R :

$$\frac{\partial u}{\partial \nu} + \alpha_R u = 0.$$

We notice that if $\alpha_R = 0$, then we obtain the homogeneous Neumann boundary condition. For these two types of conditions the trace of the solution u is not fixed on $\partial\Omega$ as it is in the case of the Dirichlet boundary condition. Thus, we consider all elements of the Hilbert space $H^1(\Omega)$ are convenient for the weak formulation. As previously, we take $f \in L^2(\Omega)$, which ensures that $-\Delta u \in L^2(\Omega)$. We know that for a d -set compact boundary $\partial\Omega$ on which it is defined a d -measure μ if we consider a trace of $u \in H^1(\Omega)$, then $\text{Tr } u \in L^2(\partial\Omega, \mu)$ (see Theorem 1.3.2). Therefore, with the help of the Green formula (1.10) we obtain the following variational problem for the homogeneous Robin boundary problem: to find $u \in H^1(\Omega)$ such that

$$\forall v \in H^1(\Omega) \quad \int_{\Omega} \nabla u \nabla \overline{v} dx + \int_{\partial\Omega} \alpha_R \text{Tr } u \overline{\text{Tr } v} d\mu = \int_{\Omega} f \overline{v} dx. \quad (1.24)$$

If $\alpha > 0$ then it is possible to show that

$$\int_{\Omega} \nabla u \nabla \overline{v} dx + \int_{\partial\Omega} \alpha_R \text{Tr } u \overline{\text{Tr } v} d\mu =: \{u, v\}$$

is an equivalent inner product on $H^1(\Omega)$ (see TD 2), and thus to apply as previously the Riesz representation theorem to deduce its well posedness.

Attention: If $\alpha = 0$, *i.e.* in the case of the Neumann homogeneous condition, $\|\nabla u\|_{L^2(\Omega)}$ is only a semi-norm on $H^1(\Omega)$:

if $u \in H^1(\Omega)$ and $\|\nabla u\|_{L^2(\Omega)} = 0$, it does not imply that $u = 0$ almost everywhere on Ω , but only that $u = \text{constant}$ (not necessary equal to zero) almost everywhere on Ω .

Thus it is not possible to apply the Riesz representation theorem, and moreover, it is easy to see for $f = 0$, that any constant on Ω solves the problem (there is no unicity).

One of the ways to avoid this ill-posedness it is to impose in a part of the boundary a Dirichlet boundary condition or a Robin boundary condition with $\alpha_R > 0$.

If $\alpha_R < 0$ then the Robin boundary problem becomes also ill-posed.

1.4.4 Poisson equation with mixed boundary conditions

In this course for the wave control applications, we specially interested in the **mixed** boundary conditions, *i.e.* when the boundary $\partial\Omega$ of the domain Ω is divided in three parts $\partial\Omega = \Gamma_{Dir} \cup \Gamma_{Neum} \cup \Gamma_{Rob}$:

1. Γ_{Dir} on which it is imposed a Dirichlet boundary condition;
2. Γ_{Neum} on which it is imposed a Neumann boundary condition;
3. Γ_{Rob} on which it is imposed a Robin boundary condition.

To be able to define the traces $\text{Tr}_{\Gamma_{Dir}} : H^1(\Omega) \rightarrow L^2(\Gamma_{Dir}, \mu)$ and $\text{Tr}_{\Gamma_{Rob}} : H^1(\Omega) \rightarrow L^2(\Gamma_{Rob}, \mu)$ separately in the same way as in Theorem 1.3.2 the trace on all boundary, we suppose each time for the convenience that

Γ_{Dir} and Γ_{Rob} are themselves compact d -sets as $\partial\Omega$ (*i.e.* they are closed subsets of $\partial\Omega$ with the same d -measure μ). We also assume that

$$\mu(\Gamma_{Neu} \cap \Gamma_{Dir}) = \mu(\Gamma_{Neu} \cap \Gamma_{Rob}) = \mu(\Gamma_{Dir} \cap \Gamma_{Rob}) = 0. \quad (1.25)$$

Remark: (*could be omitted*) If $\Gamma_{Neum} = \emptyset$, then it is still possible to modify Γ_{Dir} and Γ_{Rob} on a set of measure zero, that they will be still compacts separated by a set of zero measure. Since the trace and the boundary conditions are imposed in the sense of $L^2(\partial\Omega, \mu)$, there is no problem to say that $\partial\Omega = \Gamma_{Dir} \cup S \cup \Gamma_{Rob}$ for a set $S \subset \partial\Omega$ having $\mu(S) = 0$ on which we don't define any boundary condition.

In the case of the homogeneous mixed boundary conditions, we need have a “mixed” approach from the treatments of the homogeneous Dirichlet and Robin boundary conditions. More precisely, considering the Poisson problem for $\alpha_R > 0$

$$\begin{cases} -\Delta u = f \text{ on } \Omega, \\ u = 0 \text{ on } \Gamma_{Dir}, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{Neu}, \\ \frac{\partial u}{\partial n} + \alpha_R u = 0 \text{ on } \Gamma_{Rob}, \end{cases} \quad (1.26)$$

we need to fix the zero traces of elements of $H^1(\Omega)$ on the Dirichlet part of $\partial\Omega$ by defining

$$V(\Omega) = \{u \in H^1(\Omega) | Tr_{\Gamma_{Dir}} u = 0\}, \quad (1.27)$$

with the norm taking into account the presence of the Robin boundary condition

$$\|u\|_{V(\Omega)}^2 = \int_{\Omega} |\nabla u|^2 dx + \alpha_R \int_{\Gamma_{Rob}} |Tr_{\partial\Omega} u|^2 d\mu \quad (1.28)$$

associated as previously with the inner product

$$(u, v)_{V(\Omega)} = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx + \alpha_R \int_{\Gamma_{Rob}} Tr_{\partial\Omega} u Tr_{\partial\Omega} \bar{v} d\mu. \quad (1.29)$$

As it was previously noticed, on $V(\Omega)$, as there is a boundary part with zero traces, it holds the Poincaré inequality: there exists a constant $C_P > 0$ depending only on the diameter of Ω such that for all $u \in V(\Omega)$ it holds (1.21).

1.5 Regularity of the boundary: why it is important?

Let Ω be a bounded domain of $\mathbb{R}^n$. We have seen that in the framework of weak solutions, the Dirichlet boundary valued problem for the Poisson equation is understood in the following variational form (1.19) in which there is no more any boundary influence (to compare with (1.24)). Therefore, the unique weak solution $u \in H_0^1(\Omega)$ exists for an arbitrary bounded domain Ω by a simple application of the Riesz representation theorem.

There are two notions for the regularity question of the solution: the interior regularity and the regularity up to the boundary.

1.5.1 Interior regularity

Thanks to Evans [25] Theorem 2 p. 304 and Theorem 3 p. 316, we have (even for solutions in $H^1(\Omega)$ and thus for different boundary conditions) the interior regularity of the weak solution.

Definition 1.5.1. *Let Ω be a domain of $\mathbb{R}^n$ and K be a subset compactly included in Ω , $K \subset\subset \Omega$. The weak solution on Ω has **the interior regularity**, if it has on K the same regularity as for a domain with regular boundaries.*

For instance, in our example of the Poisson equation with homogeneous Dirichlet boundary conditions, if $f \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$ independently on the shape of $\partial\Omega$. The key point here that Ω is open.

1.5.2 Regularity up to the boundary

The property to be in $C(\overline{\Omega})$ is much more restrictive, since the continuity on a compact requires from u to be bounded and equicontinuous, and does not hold for arbitrary shapes of $\partial\Omega$ [23].

By very technical results of Nyström [58], the necessary condition for Ω is to be a non-tangentially accessible domain (a NTA domain), which we give here the definition for a possible curiosity.

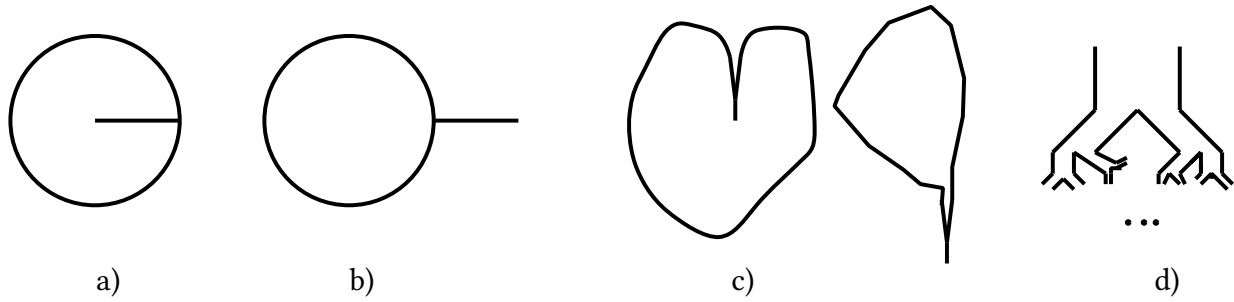


Figure 1.5 – Examples of “bad” domains, which are not NTA domains, as soon as at least one of Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ becomes infinitely thin close to the boundary. From left to right: a) a domain with incoming cusp, b) a domain with outgoing cusp, c) two types of collapsing domains, d) infinite fractal trees with parts becoming infinitely thin.

Attention: The definition of the NTA domain is not required to be read or understood. However, it is important to know that such domains exist and that they are called NTA domains. As examples, they contain domains with Lipschitz and self-similar von Koch fractal boundaries; hence they are rather general.

Definition 1.5.2 (NTA domain). [46] A bounded domain $\Omega \subset \mathbb{R}^n$ is called NTA when there exists constants M and r_0 such that:

1. *Corkscrew condition:* For any point $Q \in \partial\Omega$, $r < r_0$, there exists a point $A = A_r(Q) \in \Omega$ such that $M^{-1}r < |A - Q| < r$ and $d(A, \partial\Omega) > M^{-1}r$.
2. $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the Corkscrew condition.
3. *Harnack chain condition:* If $\epsilon > 0$ and points P_1 and P_2 belongs to Ω , $d(P_j, \partial\Omega) > \epsilon$ and $|P_1 - P_2| < C\epsilon$, then there exists a Harnack chain from P_1 to P_2 whose length depends on C and not on ϵ .

For P_1, P_2 in Ω , a Harnack chain from P_1 to P_2 in Ω is a sequence of M non-tangential balls such that the first ball contains P_1 , the last contains P_2 , and such that consecutive balls have non empty intersections. Finally, a M non-tangential ball in a domain Ω is a ball $B(A, r)$ in Ω whose distance from $\partial\Omega$ is comparable to its radius:

$$Mr > d(B(A, r), \partial\Omega) > M^{-1}r.$$

The main property of a NTA domain that Ω and its complement $\mathbb{R}^n \setminus \overline{\Omega}$ cannot be “thin” close to its boundary $\partial\Omega$.

Example 1.5.1. We give in Fig. 1.5 the examples of “bad” domains, which are not NTA domains, as soon as at least one of Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ becomes infinitely thin close to the boundary.

For C^2 boundaries it is also known the H^2 -regularity [25, Thm. 4, p. 317] of the weak solutions: if $f \in L^2(\Omega)$ and $\partial\Omega \in C^2$, then the weak solution of (1.19) $u \in H^2(\Omega) \cap H_0^1(\Omega)$ (see also Theorem 5 p. 323 [25] for higher boundary regularity).

But it is no more true in the general class of NTA domains. It is important to cite here the theorem of Nyström [58]:

Theorem 1.5.1. *Let $\Omega \subset \mathbb{R}^2$ be von Koch's snowflake (see Fig. 1.6). Let $f \in \mathcal{D}(\Omega)$ be non negative and non identically zero. Let $u \in H_0^1(\Omega)$ be the weak solution of the Poisson problem with homogeneous Dirichlet boundary condition. Then*

$$u \in H_0^1(\Omega) \cap C^\infty(\Omega) \text{ and } u \in C(\overline{\Omega}),$$

but

$$u \notin H^2(\Omega).$$

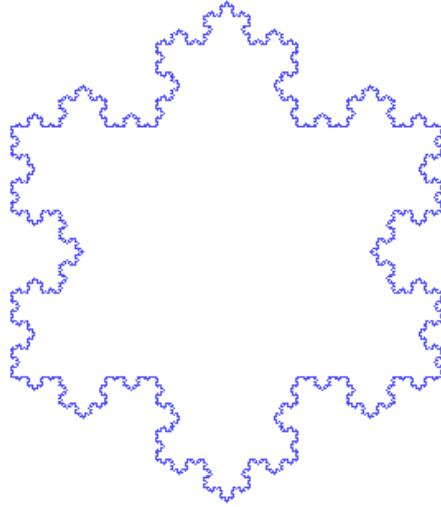


Figure 1.6 – von Koch's snowflake (5th generation). For its generation and geometrical definition see Chapter 2.

However, it holds for the convex polygonal domains [36]. The convexity condition does not allow the incoming angles, which create the singularities. This is an example treated on TD1, which we state briefly here.

Let Ω be the “camembert” (see Fig. 1.7) defined in the polar coordinates by

$$\Omega = \{(r, \theta) / r < 1, \quad 0 < \theta < \alpha\},$$

bounded by an arc of circle

$$\Gamma_0 = \{(1, \theta), \quad 0 \leq \theta \leq \alpha\}$$

and the rays

$$\Gamma_1 = \{(r, 0), \quad 0 \leq r \leq 1\} \cup \{(r, \alpha), \quad 0 \leq r \leq 1\}.$$

Let $g \in C(\Gamma_0)$ be such that $g(0) = g(\alpha) = 0$. We consider the boundary valued problem

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega, \\ u = g & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_1. \end{cases} \quad (1.30)$$

We suppose that $\alpha > \pi$. The gradient in the polar coordinates is given by

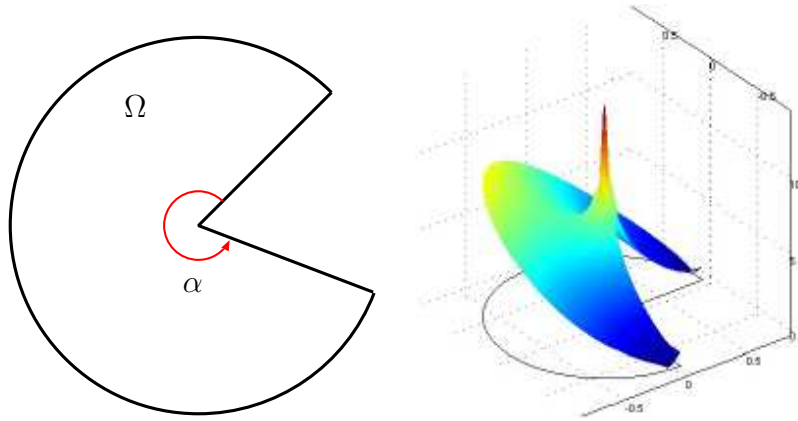


Figure 1.7 – Domain “camembert” on the left. On the right, the divergence on the coin point of the modulus of the gradient (by P. Laurent).

$$\nabla u = \left(\frac{\partial u}{\partial r}, \frac{1}{r} \frac{\partial u}{\partial \theta} \right)^t.$$

Using decomposition of the solution in a series, it is thus possible to show that

$$\nabla u = \left(\frac{\partial u}{\partial r}, \frac{1}{r} \frac{\partial u}{\partial \theta} \right)^t = \begin{pmatrix} \sum_k a_k \frac{k\pi}{\alpha} r^{\frac{k\pi}{\alpha}-1} \sin\left(\frac{k\pi}{\alpha}\theta\right) \\ \sum_k a_k \frac{k\pi}{\alpha} r^{\frac{k\pi}{\alpha}-1} \cos\left(\frac{k\pi}{\alpha}\theta\right) \end{pmatrix}$$

diverge pour $\pi < \alpha < 2\pi$ quand $r \rightarrow 0$. Moreover we show:

Proposition 1.5.1. *The point of an entering corner is a singular point of the classical solution of Laplace’s equation in that the gradient of the solution tends to infinity at that point (see Fig. 1.7 on the right). So we have*

$$u \in C(\overline{\Omega}) \cap C^2(\Omega), \quad \text{but} \quad u \notin C^1(\overline{\Omega}).$$

Chapter 2

Fractals and applications

The interest to the irregular geometries, and in particular, to fractals, initially comes from a goal to model the complexity of the natural structures of different kinds [55, 56], as the coast of Britain, human lung, intestine, the microstructure of electrodes, porous materials... The fractal models present a particular case of boundaries or objects themselves of "complicated" multi-scale structures with a bigger dimension than regular shapes. This higher dimension can model, for instance, objects with "thick" boundaries, as vascularized tumors, and be a way to create models favoring energy exchanges between two media [21, 63, 8] for the diffusion process, and [28, 61] for the wave propagation. From physical experiments [51, 26], the impact of the irregularity compared to the smooth geometry is known to be huge. It is also in the origin of the energy localization [27] explained by spectral problems (see Chapter 4). Recently, mathematically, it was shown that the irregular geometries, describing by measures with higher dimensions to compare to the regular, typically Lipschitz, boundaries, are energy minimizers. For instance, in applications of the absorbing anti-noise barriers near highways, non-Lipschitz shapes minimize the acoustical energy [43]. Non-Lipschitz shapes are also the most stables architecture structures under loads when investigating deformation of roofs under snow [44]. In addition, it is shown that the minimum of the energy generally can not be reached by Lipschitz shapes [53]. Since fractals are ideal mathematical shapes, not able to be represented numerically, the question of approximation of optimal irregular shapes by more regular ones is essential from the theoretical and numerical point of view. Irregular geometries appear, going from applications to mathematics, and also going from mathematics to applications.

2.1 Self-similar fractals

There are two types of fractals: fractal domain itself as in Fig. 2.1 and domains with a fractal boundary as in Fig. 2.2. The common to these fractals that the self-similar principle constructs them, *i.e.*, by applying to each basic element of a previous fractal generation a chosen number of contractive similitudes (transformations) to obtain the next generation of fractal (see also Fig. 2.3, for more examples and more details see [56]). As it is shown in Fig. 2.1, if we zoom on a fragment of a fractal, this fragment is the same fractal itself, and we can zoom infinitely many times, and each time we find the same shape as previously.

The paradox of the Serpinski gasket is that by its construction it has the surface equal to 0 and

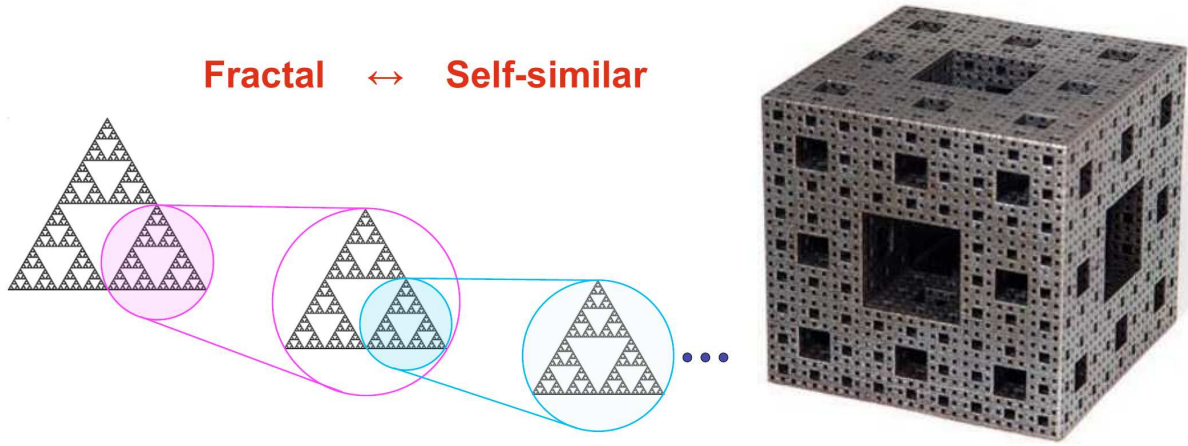


Figure 2.1 – Example of a fractal domain: Sierpinski gasket in $\mathbb{R}^2$ (fractal dimension $d = \log 3 / \log 2$) and Menger sponge in $\mathbb{R}^3$. (E. Akkerman and Wikipedia)

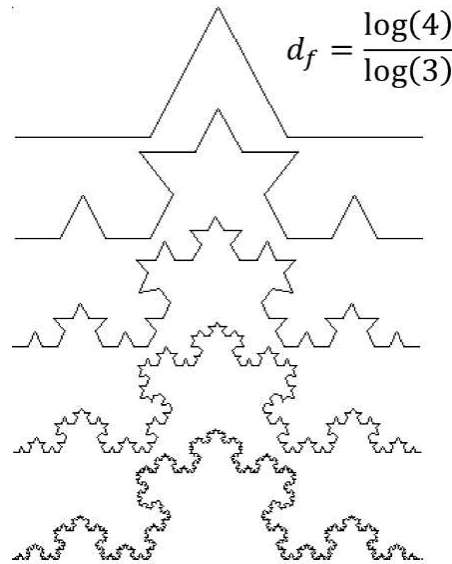


Figure 2.2 – Five first generations of von Koch's snowflake fragment of boundary (fractal dimension $1 < d = \log 4 / \log 3 \approx 1.26 < 2$). (Wikipedia)

a boundary length equal to $+\infty$:

$$S = S_0 \left(1 - \frac{1}{4} - \frac{3}{4^2} - \frac{9}{4^3} - \dots \right) = 0,$$

$$L = 3L_0 \left(1 + \frac{1}{2} + \frac{3}{2^2} + \frac{9}{2^3} + \dots \right) = +\infty,$$

but considered with its d -measure, it has a finite strictly positive measure and it is a bounded set of $\mathbb{R}^2$. Without entering in details how to find the dimension of a fractal, we can notice that each time d is a real number strictly bigger than $n - 1$ and less than n for a domain of $\mathbb{R}^n$. Roughly speaking, each time $d = \log(j) / \log(m)$, where j is equal to the number of elements/fragments in the first generation and m is equal to the number of times that the fragments of the next generation are smaller than the same fragments on the previous generation (the inverse of the contraction number). For example, for von Koch's snowflake $d = \log 4 / \log 3$. In this case, 4 is the number of elements of the first generation, and 3 corresponds to the scaling factor $1/3$ for

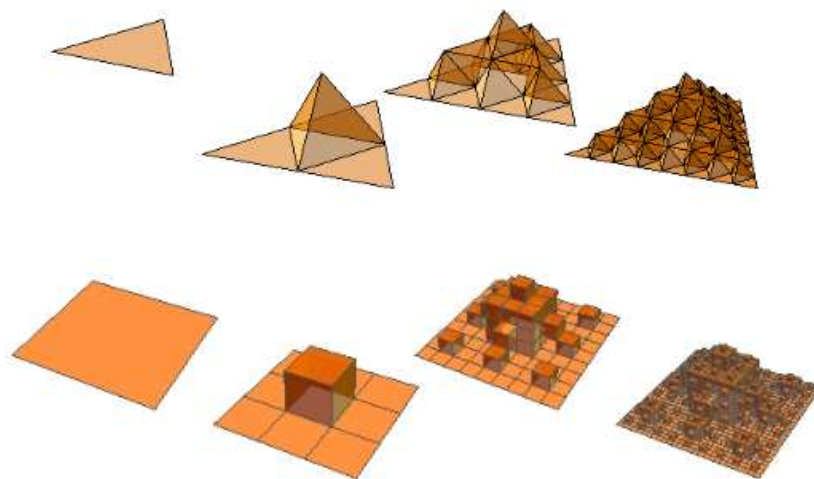


Figure 2.3 – First generations of 3D domains with fractal boundary of von Koch type. On the top, the fractal dimension $2 < d = \log(6)/\log(2) \approx 2.59 < 3$ and on the bottom the fractal dimension $2 < d = \log(13)/\log(3) \approx 2.33 < 3$. (Wikipedia)

the sizes to obtain the next generation. Formally, the dimension d is the solution of the equation $4 \left(\frac{1}{3}\right)^d = 1$.

The Sierpinski gasket and the Menger sponge of Figs 2.1 give us the examples of fractal domains, which are d -sets itself. Figs. 2.2 and 2.3 give the examples of domains with a d -set boundary, the case which we will study in detail.

Remark: There are also other examples of fractals, as the fractals created by Brownian motion or fractal trees, which for instance, could be useful to model the lungs or trees themselves. We don't consider these types of fractals in this course, as they are not helpful for our control of waves subject with applications in the noise-absorbing barriers. Besides, the fractal trees need a completely different functional analysis treatment, since they are not d -sets and not NTA domains [1].

2.2 Applications of fractals to the control of the waves

As we can notice from the previous section, all fractal's common property is to have multiscaling sizes/lengths. From physics, it is known that the interaction of a wave with a medium/obstacle depends on the geometrical shape of the medium to compare to the wavelength of the wave.

The noise (of cars, trains, planes, instruments ...) presents a source containing a complex combination of different acoustical frequencies corresponding to different wavelengths. It is complicated to model it precisely, but it is simpler to define the ranges of frequencies and wavelengths possibly, or even better, the most frequently emitted by the source. Thus the problem is to absorb the noise created by a chosen source in the most efficient way, hence on all frequency range

of the interest. As the optimality of the absorption depends on the ratio

$$\frac{\text{geometrical size of a fragment of an absorbing wall}}{\text{wavelength of the wave}} \approx \frac{1}{2}, \quad (2.1)$$

for large frequency ranges we need to have different geometrical lengths in the shape of the absorbing wall and to privilege the multiscaling geometries. Therefore, ideally, if we want to be optimal (or as close as possible to the optimal) in an infinite range of frequencies, we need to have infinitely many geometrical scaling. Hence this could be naturally modeled by a fractal boundary. However, the audible frequencies don't go to infinity, and hence practically a pre-fractal shape could be rather efficient too. On Fig. 2.4, we can see the multiscaling forms of absorbing material in the acoustical and electromagnetic anechoic chambers constructed to dissipate a wide range of frequencies. The acoustical anechoic chamber of Microsoft has $-20dB$ noise level and can be considered as the quietest place on earth. We also have an example in Fig. 2.5 of the most efficient between existing acoustical road anti-noise walls "Fractal wall"(TM), designed by B. Sapoval and M. Filoche, patent École Polytechnique and Colas, created from the cement-wood porous material with a multiscale shape (see Subsection 2.3). For a more detailed discussion about the control of the waves by the shape of a wall, see Chapter 5.

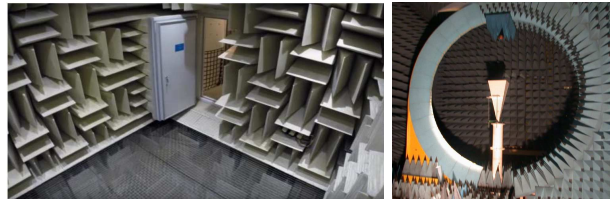


Figure 2.4 – On the left: the anechoic chamber of Microsoft: $-20dB$ noise level (Alpha Acoustics). On the right: the electromagnetic anechoic chamber of Thales (D. Lecoint).



Figure 2.5 – "Fractal wall" (TM) Brevet Colas-École Polytechnique 03/00881
US patent 7308965B2.

From another point of view, to absorb the energy of an acoustical wave, the barrier or the wall need to be constructed with a porous medium (see for examples Fig. 2.6), which by its complicity could be modeled as a fractal domain, for instance as Menger sponge. Nevertheless, we prefer to avoid this additional difficulty, knowing by the experimental measurements on a chosen porous material its essential macroscopic properties, from which it is possible to define the corresponding energy losses (see Chapter 3).

2.3 Example of "Fractal" TM wall

For an introduction to experimental acoustical framework of auto traffic noise see section 2 of [4].

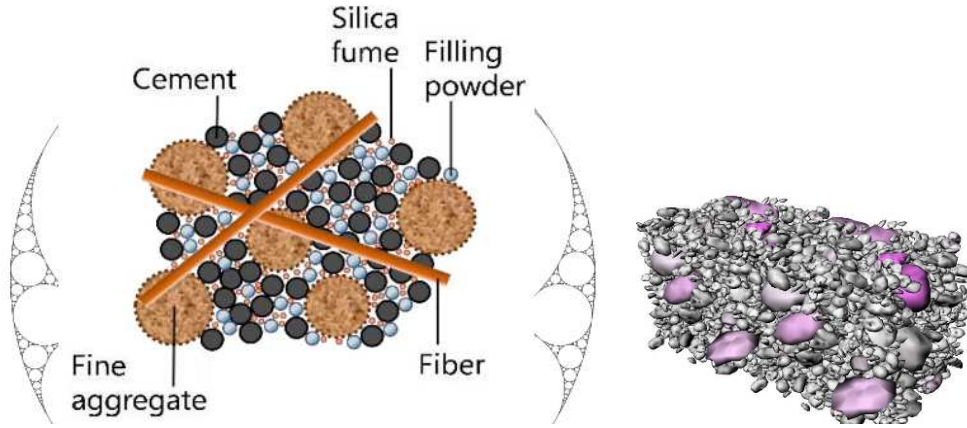


Figure 2.6 – Fractals as multiscales porous materials (D. Grebenkov).

Ideas of creation of a bigger interaction of the wave with the absorbent media by a multiscale geometry favored the eigenmodes localization phenomena [27] were successfully applied in the construction of the anti-noise barrier named Fractal Wall¹ by B. Sapoval and M. Filoche (PMC, CNRS, École Polytechnique), the most efficient wall in the absorption of low-frequency traffic of highways (see Fig. 2.5). It is made with a porous material cement with wood inclusions and has a multiscale shape to capture the lower frequency waves. The construction of one panel performs by demolding. The demolding procedure of small fragments of the wall is delicate, as they can be easily broken. This complication of its construction makes the Fractal Wall expensive compare to classical, less efficient walls. In Fig. 2.7, it is shown the coefficient of the absorption of the Fractal Wall to compare to the wall with the plane shape. This coefficient is measured experimentally (in a reverberant chamber). It is denoted here by α , but it is not the complex coefficient in the Robin boundary condition of (5.7).

Considering the decibel performances of the wall, it is possible to introduce the efficiency coefficient by EU norm 20354. Discretizing the range of frequencies and taking the average of the traffic noise intensity in these frequencies, the efficiency coefficient is defined by the formula:

$$DL_{\alpha} = -10 \log_{10} \left(1 - \frac{\sum_i \alpha_i I_i^{(A)}}{\sum_i I_i^{(A)}} \right) = 17dB(A).$$

The best wall in the European market before had $11dB(A)$.

2.4 PDEs on domains with fractal boundaries

As it was mentioned in Chapter 1, for the simplicity, to avoid too much abstract framework, in this course we only suppose that

1. for $d = n - 1$ the boundary $\partial\Omega$ is Lipschitz (see Definition 1.3.2);
2. for $d \in]n - 1, n[$ the boundary $\partial\Omega$ is a self-similar fractal (for examples see Section 2.1).

For the treatment of more general boundaries, see [5, 62, 53, 43, 44].

There is the following important theorem, which is admitted without proof, but it is very important to know, as we will use it many times in what following:

¹product of Colas Inc., French patent N0- 203404; U.S. patent 10"508,119.

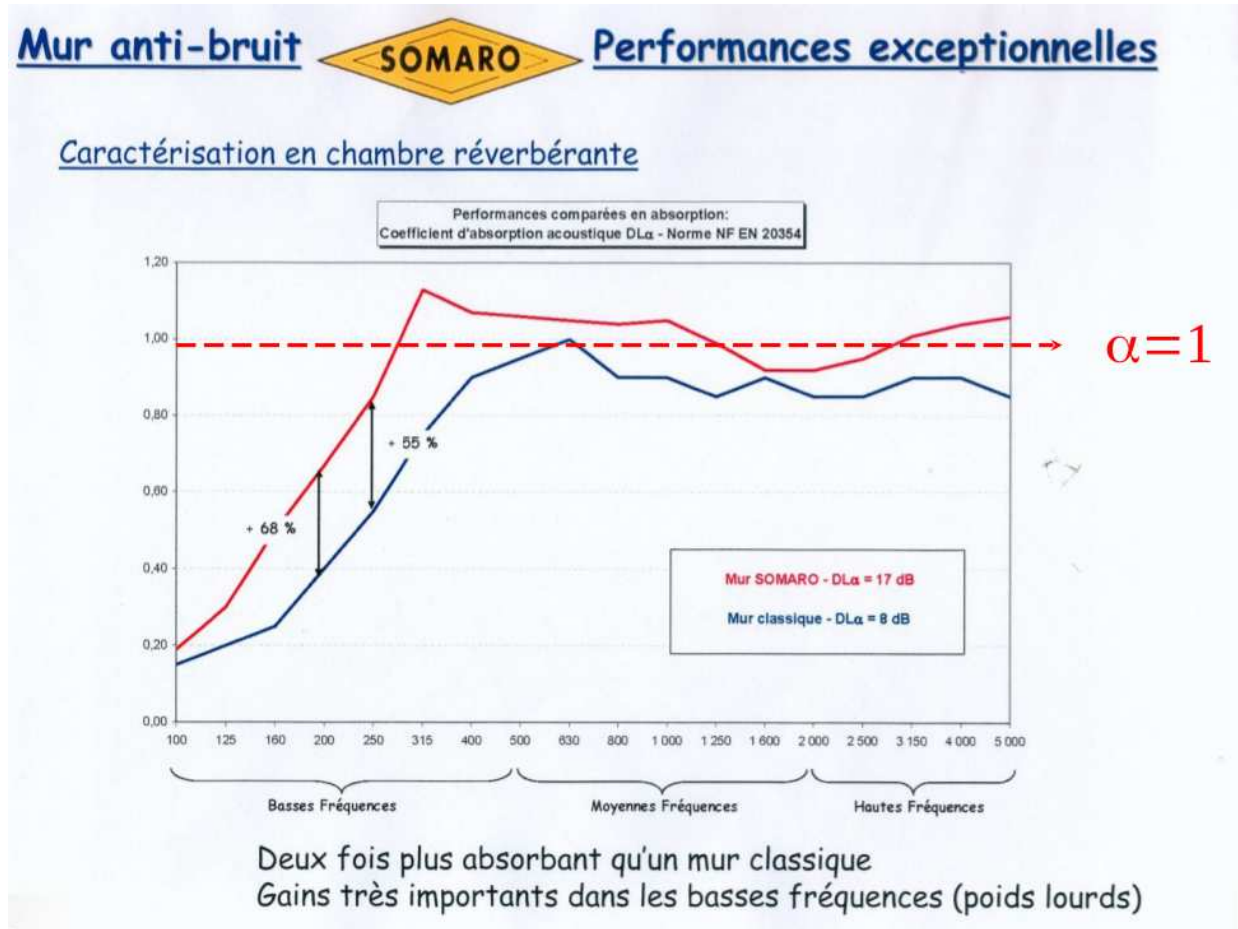


Figure 2.7 – B. Sapoval: The experimental coefficient of absorption (measured in a reverberant chamber) of anti-noise barrier 'Fractal Wall' TM, Colas-École Plytechnique (French patent N0-203404; U.S. patent 10"508,119), compared with the plane wall of the same volume and made of the same porous material (cement with wood inclusions). The main performance of the fractal wall is in the lower frequencies.

Theorem 2.4.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with a d -set compact boundary $\partial\Omega$, Lipschitz for $d = n - 1$ or self-similar fractal for $d \in]n - 1, n[$, on which it is defined a d -measure μ , $n - 1 \leq d < n$. Then,

1. $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$;
2. $\text{Tr}_\Omega : H^1(\mathbb{R}^n) \rightarrow H^1(\Omega)$ is a linear continuous and surjective operator with linear bounded right inverse (the extension operator $E_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$);
3. the operators $\text{Tr} : H^1(\mathbb{R}^n) \rightarrow L^2(\partial\Omega, \mu)$, and $\text{Tr}_{\partial\Omega} : H^1(\Omega) \rightarrow L^2(\partial\Omega, \mu)$ are linear compact operators;
4. the usual norm $\|\cdot\|_{H^1(\Omega)}$ is equivalent to

$$\|u\|_{\text{Tr}} = \left(\int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} |\text{Tr} u|^2 d\mu \right)^{\frac{1}{2}} \quad (2.2)$$

for all $u \in H^1(\Omega)$.

We will prove the equivalence of the norms on TD 2. This theorem is useful to establish the well-posedness of the Poisson or the Helmholtz problems with Robin or mixed boundary conditions

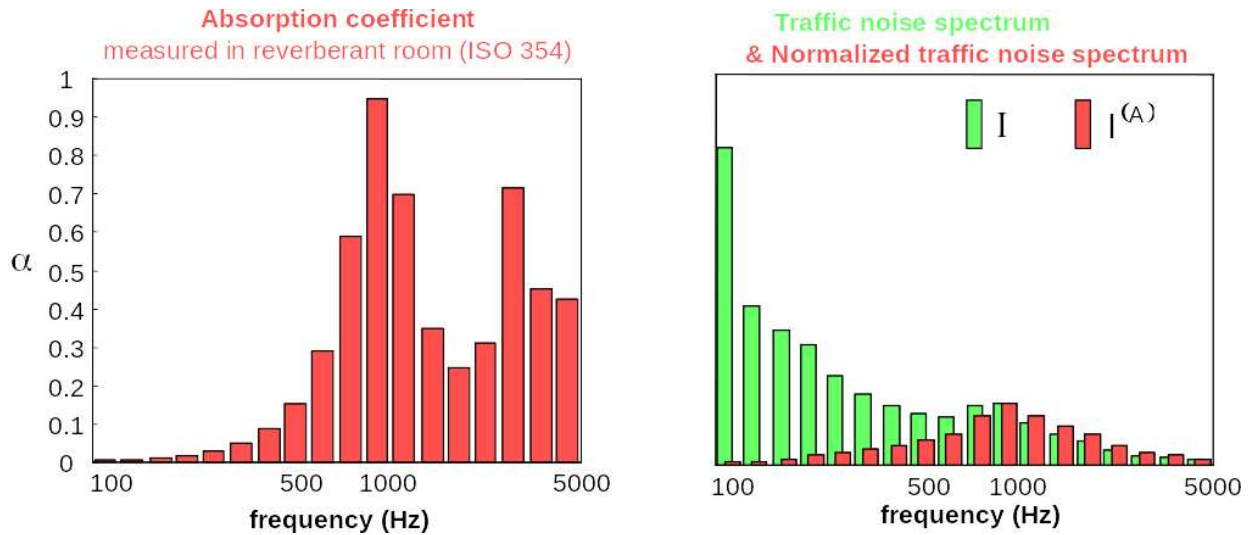


Figure 2.8 – On the left the coefficient of the absorption of the Fractal Wall and on the right the traffic noise spectrum and the normalized traffic noise spectrum (B. Sapoval).

on d -sets. But before to do it, we need to understand what it is a compact set and a compact operator, explained in Chapter 6.

As mentioned previously, the fractal boundary allows all notions as the normal derivative and the integration by parts only in the weak sense.

Chapter 3

Models of wave propagation

In this course, we chose the simplest possible models of the wave propagation useful for our typical acoustic problem: to absorb most efficiently the noise of a source by a wall of a fixed porous material (and of fixed quantity). We notice that the porous materials are acoustical absorbent. For the electromagnetic waves, the absorbent materials are different (ferromagnetic foams) because of the different nature of the wave, and they were presented in the introductory part of this course. The “efficient absorption” is supposed to be ensured by a wide choice of wall’s shape, which acoustical performances are considered in the comparison of a plane wall with the same quantity of the porous material (of the same volume). This problem is a complex physical problem schematically presented in Fig. 3.1. We will discuss it in detail for an example

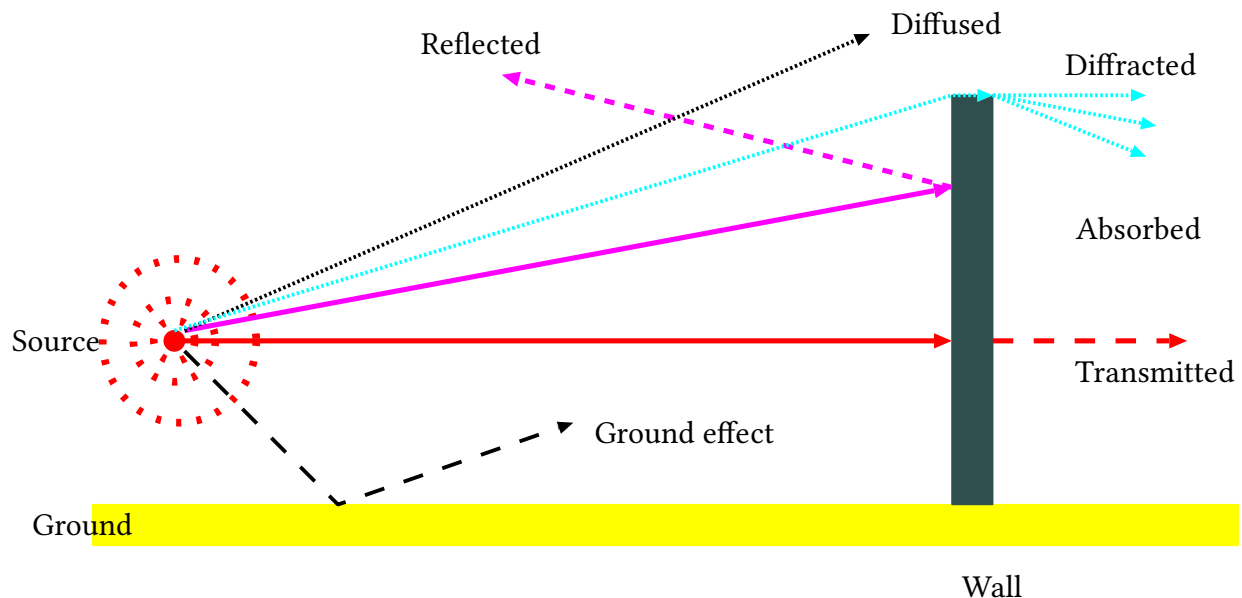


Figure 3.1 – The complexity of the wave (acoustical or electromagnetic) propagation problem to model.

in the acoustical framework.

3.1 Wave equation and Helmholtz equations

3.1.1 Linear wave equation

Let us say several words about the derivation of the linear wave equation (how it could be obtained).

For instance, it could be obtained by a linearization (by performing small perturbations around a constant state with the choice to keep only the largest terms) of a very general thermo-elastic dynamical system given by compressible Navier-Stokes equations with the assumption [60] of the potential motion and initially small viscosity effects of the medium.

For simplicity, let us demonstrate the derivation of the wave equation from the isentropic (*i.e.* independent of entropy) evolution of the thermo-elastic non-viscous media described by the following Euler system:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad \rho(\partial_t v + v \cdot \nabla v) = -\nabla p(\rho). \quad (3.1)$$

Here the ρ is the density, v is the velocity and p is the pressure, supposed to be a function of ρ . Any constant state (ρ_0, v_0) is a stationary solution of system (3.1). Linearization near this state introduces the perturbations with the same order ϵ (a dimensional parameter much smaller than 1)

$$\rho = \rho_0 + \epsilon \tilde{\rho}, \quad v = v_0 + \epsilon \tilde{v}.$$

Taking $v_0 = 0$ and putting the last expressions in (3.1), we obtain the acoustic system:

$$\partial_t \tilde{\rho} + \rho_0 \nabla \cdot \tilde{v} = 0, \quad \rho_0 \partial_t \tilde{v} + p'(\rho_0) \nabla \tilde{\rho} = 0. \quad (3.2)$$

System (3.2) is equivalent to the wave equation:

$$\frac{1}{c^2} \partial_t^2 \tilde{\rho} - \Delta \tilde{\rho} = 0, \quad \partial_t \tilde{v} = -\frac{p'(\rho_0)}{\rho_0} \nabla \tilde{\rho}, \quad (3.3)$$

where $c = \sqrt{p'(\rho_0)}$ is the speed of sound in the unperturbed media. The linear wave equation can be also obtain for the velocity or for the pressure of the wave.

Actually, the linear wave equation appears from (3.1) if we consider only the terms of the zero order on ϵ :

$$\partial_t^2 u - c^2 \Delta u = 0. \quad (3.4)$$

The second partial time derivative $\frac{\partial^2 u}{\partial t^2}$ in the wave equation requires **two initial data**:

$$u|_{t=0} \stackrel{\text{note}}{=} u(0, x) = u_0(x), \quad x \in \Omega \quad (3.5)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} \stackrel{\text{note}}{=} \frac{\partial u}{\partial t}(0, x) = u_1(x), \quad x \in \Omega \quad (3.6)$$

where u_0 and u_1 are given functions of x which describe the initial wave propagation in time $t = 0$. Without the initial data u_0 and u_1 the propagation process is undefined.

Remark: Compare the difference between the solution of the ordinary differential equation of the second order

$$\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = e^x$$

and of its Cauchy problem

$$\begin{aligned} \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y &= e^x, \\ y(0) &= 0, \\ \frac{dy}{dx}(0) &= 0. \end{aligned}$$

Problem (3.4)–(3.6) posed for $x \in \mathbb{R}^n$ is called **the Cauchy problem** for the wave equation.

But if we search a solution of the problem in a domain $\Omega \subset \mathbb{R}^n$ with a boundary $\partial\Omega \neq \emptyset$, we need to identify the **boundary conditions** as well (see Subsection 1.4.1).

In the acoustical wave propagation framework, the boundary conditions of Subsection 1.4.1 considered for the linear wave equation could have the following physical interpretation:

1. the Dirichlet boundary condition $u|_{\Gamma_{Dir}} = \phi(t, x)$, $x \in \Gamma_{Dir}$, $t \geq 0$ models the source of a sound or noise projected to the boundary Γ_{Dir} ;
2. the homogeneous Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\Gamma_{Neu}} = 0$ for $t \geq 0$ models the reflective surfaces or sometimes the surfaces with very small and negligible absorption.
3. the homogeneous Robin boundary condition $\frac{\partial u}{\partial \nu} + \alpha u|_{\Gamma_{Rob}} = 0$ models more precisely the reflective/absorption phenomena depending on the values of the parameter α . We will consider this boundary condition in the next section.

If we would like to model the linear wave propagation in an open or a partially open situation as in Fig. 3.1, then especially for numerics, we need to limit the computations of the solution to a bounded domain. In this situation, we need to impose **artificial boundary** with some absorbent boundary conditions, which are maximum possibly non-reflective. If it is the case, the wave, absorbed by this artificial boundary, is not reflected and hence does not affect any propagation changes inside the domain. Thus, the obtained solution inside the bounded domain can be considered by the desired solution of the initial open space problem, since the main property, to have outgoing waves never returning back, is (at least approximatively) respected.

There is a lot of literature, theoretical and numerical, on this subject. Numerically, it is sometimes more convenient to create not an artificial boundary, but an artificial region or a part of the domain, called Perfect Matched Layers (PML) on which the wave is (artificially) exponentially attenuated (see, for instance, the references given in the introduction of this article [52]).

In this course, we mainly consider the damping of the waves modeling by the Robin type boundary condition and, in particular, the artificial boundary conditions for the Helmholtz equation named the Sommerfeld radiation conditions for the scattering problems.

The problem to find u satisfying for all $x \in \Omega$ and $t \geq 0$ the linear wave equation (3.4), the initial conditions (3.5)–(3.6) and some boundary conditions discussed above is called the **initial-**

boundary valued problem for the wave equation.

Let us just consider for an example the homogeneous Dirichlet initial-boundary valued problem for some initial data $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$.

Multiplying formally the linear wave equation (3.4) by $\partial_t u$ and integrating over Ω , we have

$$\int_{\Omega} (\partial_t^2 u - c^2 \Delta u) \partial_t u dx = 0.$$

To be able to integrate by parts using Proposition 1.3.1 we need to have $u \in H_0^1(\Omega)$ and Δu in $L^2(\Omega)$. For the well posedness of $\int_{\Omega} \partial_t^2 u \partial_t u dx$ we also need to have $\partial_t^2 u$ and $\partial_t u$ in $L^2(\Omega)$.

Thus, for the simplicity we suppose that $u \in C^2([0, T], \hat{H}_0^1(\Omega))$ with the notation

$$\hat{H}_0^1(\Omega) = \{u \in H_0^1(\Omega) \mid \Delta u \in L^2(\Omega)\}.$$

Remark: To say $\Delta u \in L^2(\Omega)$ for $u \in H_0^1(\Omega)$, it means to say that there exists an element $f \in L^2(\Omega)$ such that $\Delta u = f$ (in the distributional sense of $\mathcal{D}'(\Omega)$). As $u \in H_0^1(\Omega)$, it means that u is the weak solution of the Poisson problem with the homogeneous Dirichlet boundary condition:

$$\forall v \in H_0^1(\Omega) \quad \int_{\Omega} \nabla u \nabla v dx = \int_{\Omega} f v dx.$$

It is known, by the Lax-Milgram Theorem or the Riesz Representation Theorem, that for all $f \in L^2(\Omega)$ there exists a unique solution $u \in H_0^1(\Omega)$. Consequently, the mapping

$$f (= \Delta u) \in L^2(\Omega) \mapsto \text{the weak solution } u \in H_0^1(\Omega)$$

is bijective.

Thus, the space $\hat{H}_0^1(\Omega)$ with the norm $\|u\|_{\hat{H}_0^1(\Omega)}^2 = \|u\|_{H_0^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2$ is a (closed!) subspace of $H_0^1(\Omega)$ of all weak solutions of the homogeneous Dirichlet Poisson problem.

Remark: If $w \in \hat{H}_0^1(\Omega)$ for $\Omega \subset \mathbb{R}^n$, $n \geq 2$, then it does not follow that $w \in H^2(\Omega)$ or $w \in H_0^2(\Omega)$. The existence of w , its first order weak derivatives and its weak Laplacian $\partial_{x_1}^2 + \dots + \partial_{x_n}^2$ in $L^2(\Omega)$ is not enough to ensure in addition the existence of all weak derivatives of the second order in $L^2(\Omega)$. Indeed,

$$H_0^2(\Omega) \subsetneq \hat{H}_0^1(\Omega).$$

The space $\hat{H}_0^1(\Omega)$ is the natural space to provide the validity of the Green formula and formula of integration by parts. Hence, it is a natural space [22] to consider the weak solutions of different models involving the Laplacian in general domains without a regular boundary, which could ensure H^2 -regularity of the weak solutions of the Poisson boundary value problem.

In this case, we can apply the Green formula on the term with the Laplacian and also change the integration over Ω and the time derivative (by the Theorem of the derivation under the integral from CIP and EDP courses of the first year).

Indeed, by the integration by parts, and taking into account that $\partial_t u \in H_0^1(\Omega)$ (i.e. its trace on

$\partial\Omega$ is zero), we firstly have

$$\int_{\Omega} \partial_t u \partial_t^2 u \, dx + \int_{\Omega} c^2 \nabla u \cdot \partial_t \nabla u \, dx = 0.$$

Then we notice that

$$\partial_t u \partial_t^2 u = \partial_t \frac{|\partial_t u|^2}{2} \quad \text{and} \quad \nabla u \cdot \partial_t \nabla u = \partial_t \frac{|\nabla u|^2}{2}.$$

Consequently, writing

$$\int_{\Omega} \partial_t \frac{|\partial_t u|^2}{2} \, dx = \frac{d}{dt} \int_{\Omega} \frac{|\partial_t u|^2}{2} \, dx, \quad (3.7)$$

$$\int_{\Omega} c^2 \partial_t \frac{|\nabla u|^2}{2} \, dx = \frac{d}{dt} \int_{\Omega} c^2 \frac{|\nabla u|^2}{2} \, dx, \quad (3.8)$$

we conclude that

$$\frac{d}{dt} \left(\int_{\Omega} [|u_t|^2 + c^2 |\nabla u|^2](t, x) \, dx \right) = 0.$$

We recognize in this quantity the total energy of the wave problem given by the sum of the kinetic and potential energy:

$$E(t) = \int_{\Omega} [|u_t|^2 + c^2 |\nabla u|^2](t, x) \, dx. \quad (3.9)$$

Therefore, we obtained the conservation of the energy of our initial-boundary valued problem with the homogeneous Dirichlet boundary condition:

$$\frac{d}{dt} E(t) = 0.$$

Let us just notice that this equivalent to

$$\forall t \geq 0 \quad E(t) = E(0) = \int_{\Omega} [|u_1|^2 + c^2 |\nabla u_0|^2](x) \, dx.$$

Question: Do we have a conservation of the energy if we replace the Dirichlet condition by the homogeneous Neumann boundary condition?

What will change if we consider the homogeneous Robin boundary condition with $\alpha > 0$? What is happening in the case when $\alpha < 0$?

3.1.2 Helmholtz equation

Let us consider a non homogeneous wave equation

$$\partial_t^2 u - c^2 \Delta u = f(x, t)$$

with a source term f periodic with a frequency ω on time and with the spatial amplitude $c^2 f(x)$,

$$f(x, t) = c^2 f(x) e^{-i\omega t}.$$

Therefore, if we search the solutions in the form $u(x, t) = U(x) e^{-i\omega t}$ with the unknown amplitude $U(x)$, then for the function $U(x) \stackrel{\text{note}}{=} u(x)$ we obtain a stationary equation

$$\Delta u + k^2 u = -f(x), \quad \text{with the wave number} \quad k = \frac{\omega}{c}. \quad (3.10)$$

Equation (3.10) is called the Helmholtz equation.

Sommerfeld radiation conditions for the scattering problems

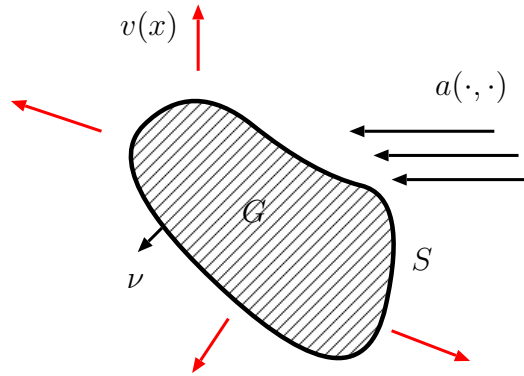


Figure 3.2 – Example of the bounded obstacle G with a boundary S .

An example of the boundary-valued problem for the Helmholtz equation is the scattering problem.

For $x \in \mathbb{R}^2$ or $\mathbb{R}^3$, suppose that the incoming (from infinity) plane wave $e^{ik(a,x)}$, $|a| = 1$, $k > 0$ is modified due to the presence of some obstacles on the boundary S of the bounded region G (Fig. 4.5). The obstacle can be defined, for example, using the **boundary condition** (see Subsection 1.4.1)

$$u|_S = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} \Big|_S = 0.$$

This obstacle generates a scattered wave $v(x)$. Far from the scattering centers this wave will be close to the outgoing spherical wave

$$v(x) = f\left(\frac{x}{|x|}\right) \frac{e^{ik|x|}}{|x|} + o\left(\frac{1}{|x|}\right).$$

Therefore, for $|x| \rightarrow \infty$ the wave $v(x)$ must satisfy the following conditions

$$v(x) = O\left(\frac{1}{|x|}\right), \quad \frac{\partial v(x)}{\partial |x|} - ikv(x) = o\left(\frac{1}{|x|}\right), \quad (3.11)$$

called Sommerfeld radiation conditions. The function $f(s)$, $s = \frac{x}{|x|}$, is called the amplitude of the scattering. The total wave perturbation u outside of G (in $\mathbb{R}^n \setminus G$) is given by the sum of the plane and the scattered waves:

$$u(x) = e^{ik(a,x)} + v(x).$$

3.2 Absorption phenomena

As it was mentioned, the porous materials are acoustical absorbents.

3.2.1 Porous materials

We give several examples of porous or fibrous materials which are used in the interior isolation of buildings on Fig. 3.3. As it is could easily seen from these examples (see also Chapter 2) that

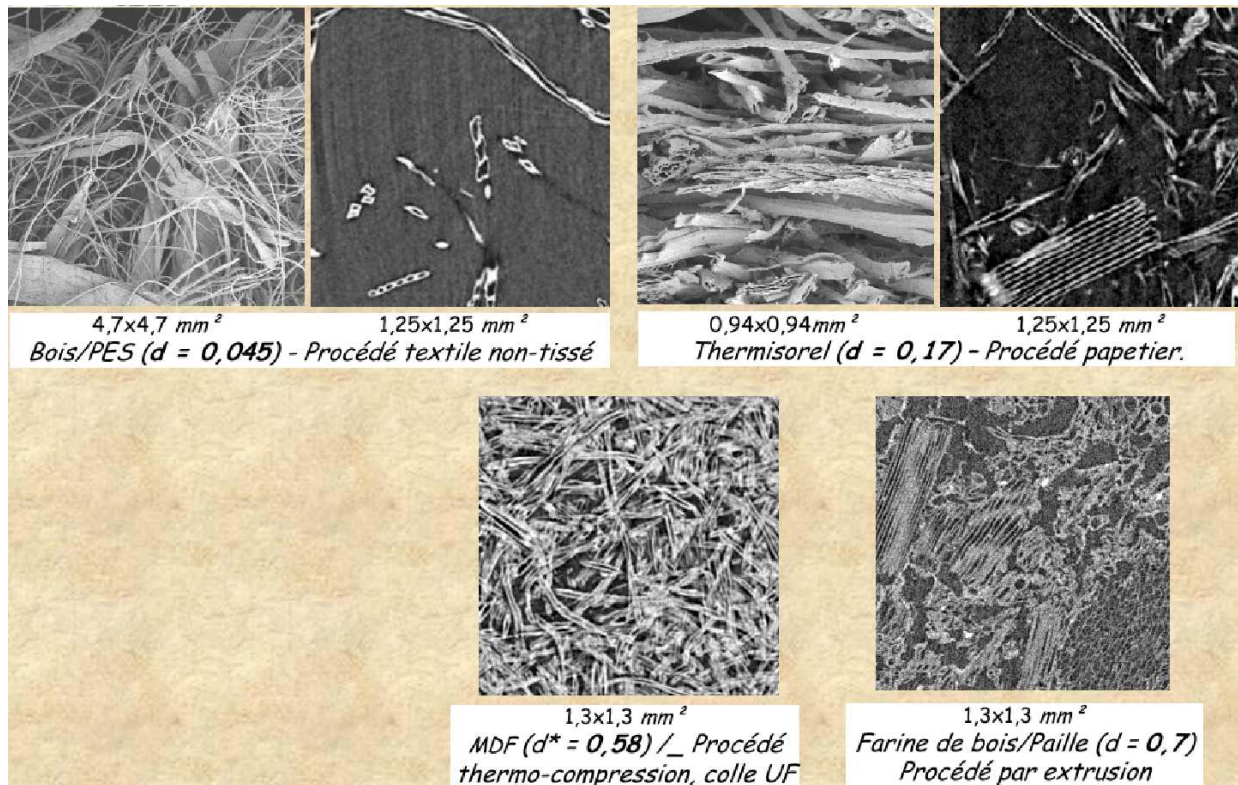


Figure 3.3 – Example of heterogeneous and anisotropic porous or fibrous materials which are used in the interior isolation of buildings (ANR “Silent wall”).

the porous materials have very complicated micro structure filled with the air. The complexity of a porous material and its dissipative properties can be described by these physical macroscopic characteristics:

1. the porosity ϕ (dimensionless),
2. the tortuosity α_h (dimensionless)
3. the resistivity to the passage of air σ ($N.m^{-4}.s$, i.e. depends on the size of the tested material).

The experimental data for the fibrous materials given partially on Fig. 3.3, named Isorel, ITFH, B4 and B5 of these physical characteristics are given on Fig. 3.4 and their experimental absorption coefficients are presented on Fig. 3.5.

Question: How depends the coefficient of absorption for a fixed porous materials from the thickness of a sample? How it could explained? Which material could be more interesting in the sound absorption problem and why (give different scenarios with an argumentation).

3.2.2 Damping in volume

There are two possibilities to describe the acoustic wave absorption by a porous medium. The first one is to consider wave propagation in two media, typically air and a wall, which corresponds to damping in the volume. The most common mathematical model for this is the damped

	épaisseur	résistivité	incertitude	porosité (*)	incertitude	tortuosité	long. visqueuse
matér. ISOREL	24,0	142300,0	4,2	0.70 ??	10,0	1,15	18
éch. 1	25,0	143300,0	4,2				
éch. 2	23,5	149400,0	4,0				
éch. 3	23,5	134300,0	4,5				
matériau ITFH	11,5	9067,0	4,5	0.94 ??		1	166,5
éch. 1	11,7	7040,0	5,7			1	165,5
éch. 2	11,2	10064,0	4,0			1	166,4
éch. 3	11,2	10098,0	4,0			1	167,6
matériau B4				0.10 ??	estimation	1.44	6.65
Ech. B4-1	9,5	X					
Ech. B4-2	9,5	-					
Ech. B4-3	9,5	-					
matériau B5				0.20 ??	1	1.22	13,3
Ech. B5-1	9,5	2124000,0	47,1				
Ech. B5-2	9,5	-					
Ech. B5-3	9,5	-					

Figure 3.4 – Physical macroscopic characteristics σ , ϕ , α_h of four types of porous materials for samples of different thickness (LAUM, ANR “Silent wall”).

wave equation [7]. The second one is to consider only one lossless medium, air, and to model energy dissipation by a damping condition on the boundary. In both cases, we need to ensure the same order of energy damping corresponding to the physical characteristics of the chosen porous medium as its porosity ϕ , tortuosity α_h , and resistivity to the passage of air σ [38].

Thanks to Ref. [38], we can define the coefficients in the damped wave equation (damping in volume) as functions of the characteristics mentioned above. More precisely, for a regular bounded domain $\Omega \subset \mathbb{R}^2$ (for instance $\partial\Omega \in C^1$) composed of two disjoint parts $\Omega = \Omega_{air} \cup \Omega_{wall}$ of two homogeneous media, the air in Ω_{air} and a porous material in Ω_{wall} , separated by an internal boundary Γ , we consider the following boundary value problem (for the pressure of the wave)

$$\begin{cases} \xi(x)\partial_t^2 u + a(x)\partial_t u - \nabla \cdot (\eta(x)\nabla u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n}|_{\mathbb{R}^2 \times \partial\Omega} \equiv 0, \quad [u]_\Gamma = [\eta\nabla u \cdot n]_\Gamma = 0, \\ u|_{t=0} = u_0 \mathbb{1}_{\Omega_{air}}, \quad \partial_t u|_{t=0} = u_1 \mathbb{1}_{\Omega_{air}}, \end{cases} \quad (3.12)$$

with $\xi(x) = \frac{1}{c_0^2}$, $a(x) = 0$, $\eta(x) = 1$ in air, *i.e.*, in Ω_{air} , and

$$\xi(x) = \frac{\phi\gamma_p}{c_0^2}, \quad a(x) = \sigma \frac{\phi^2\gamma_p}{c_0^2\rho_0\alpha_h}, \quad \eta(x) = \frac{\phi}{\alpha_h}$$

in the porous medium, *i.e.*, in Ω_{wall} . The external boundary $\partial\Omega$ is supposed to be rigid, *i.e.*, Neumann boundary condition are applied, and on the internal boundary Γ we have no-jump conditions on u and $\eta\nabla u \cdot \nu$, where ν denotes the normal unit vector to Γ . Here, c_0 and ρ_0 denote respectively the sound velocity and the density of air, whereas $\gamma_p = 7/5$ denotes the ratio of specific heats.

If we repeat the procedure of finding the energy of the wave equation for the damped wave equation, we obtain

$$\frac{d}{dt}E(t) = -2 \int_{\Omega_{wall}} a(x)|\partial_t u|^2 dx < 0,$$

as soon as $a(x) > 0$ in Ω_{wall} . This ensures that the energy decays in time. We can notice that if there is no porous material, *i.e.* $a \equiv 0$ on Ω , then as previously, we have the energy conservation. The question about is it possible to maximize the speed of the energy decay to have, for instance, exponential decay of energy in time, is actually the question of the theory of control of the waves. For a constant volume of the wall, constructed with a fixed porous material, the maximization

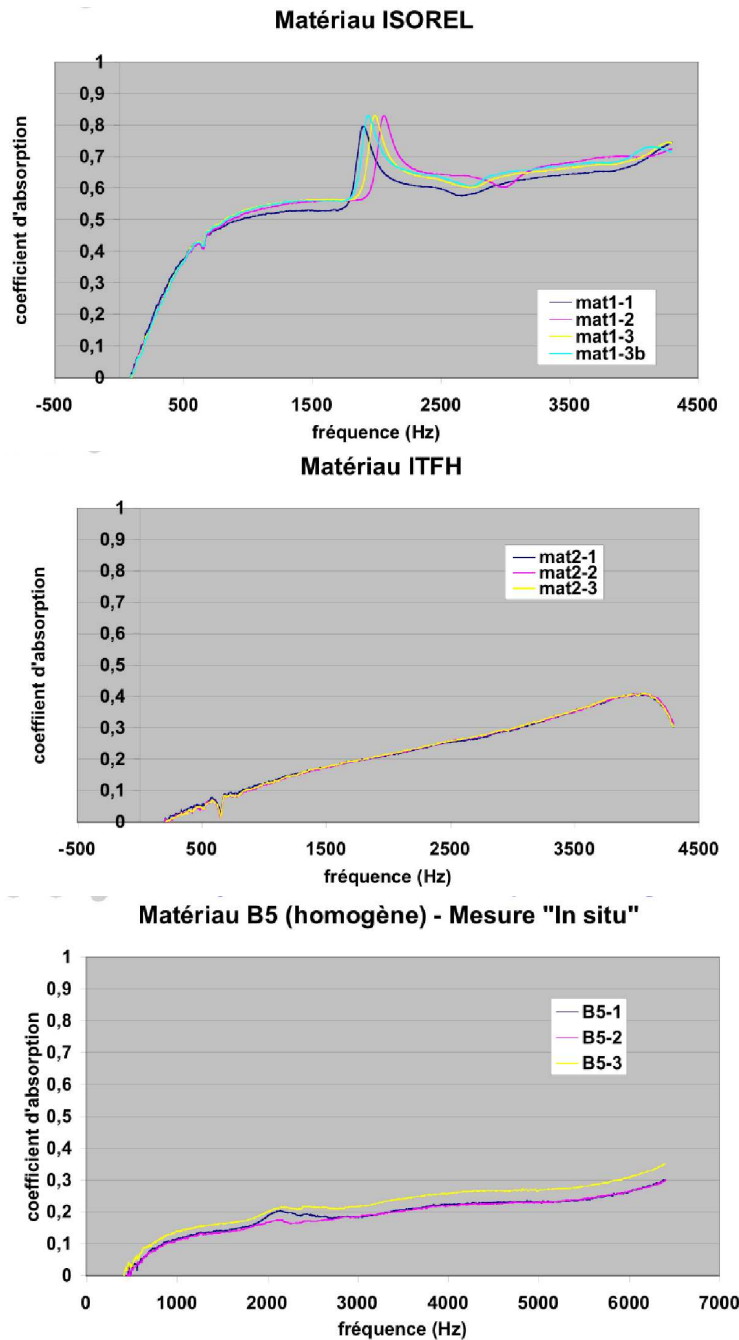


Figure 3.5 – Coefficient of absorption experimentally measured for samples of different thickness, from top to bottom, for Isorel, for ITFH and for B5.

of the energy decay depends exclusively on the shape of the wall and its location corresponding to the wave's source. These questions are discussed in Chapter 5.

This model is obtained in the assumption that the wave all times propagates in the air, involving the air in the porous medium's pores. The damped wave equation is the simplest model of this propagation, and there are other models known to be more precise but more complicated too [38, 32].

Time dependent energy decay We consider the three cavities $\Omega = \Omega_0 \sqcup \Omega_1 =]0, 1[\times]-2, 2[$, partially shown on Fig. 3.6 with two homogeneous media, air (lower part) and a porous material (upper part), separated by an internal boundary Γ_i , $i = 0, 1, 2$. To preserve the volume of each medium and to model the increasing irregularity of the interface, as compared to the plane Γ_0 (at $y = 0$), we choose Γ_1 and Γ_2 as the first two fractal generations of a symmetric element. The external boundary $\partial\Omega$ is supposed to be perfectly rigid (Neumann boundary condition). Air is considered as a loss-less medium, and the porous medium (ISOREL) is considered as a dissipative homogeneous medium. As it was mentioned in Chapter 3, using the ideas of Hamet [38], we can describe the wave propagation in the porous material by a damped wave equation involving the physical characteristics of the material: the porosity ϕ , the tortuosity α_h , and the resistivity to the passage of air σ . Denoting by c_0 and ρ_0 the sound velocity and the density in the air, and by $\gamma_p = 7/5$ the ratio of specific heats, the equations of wave propagation in Ω are given by

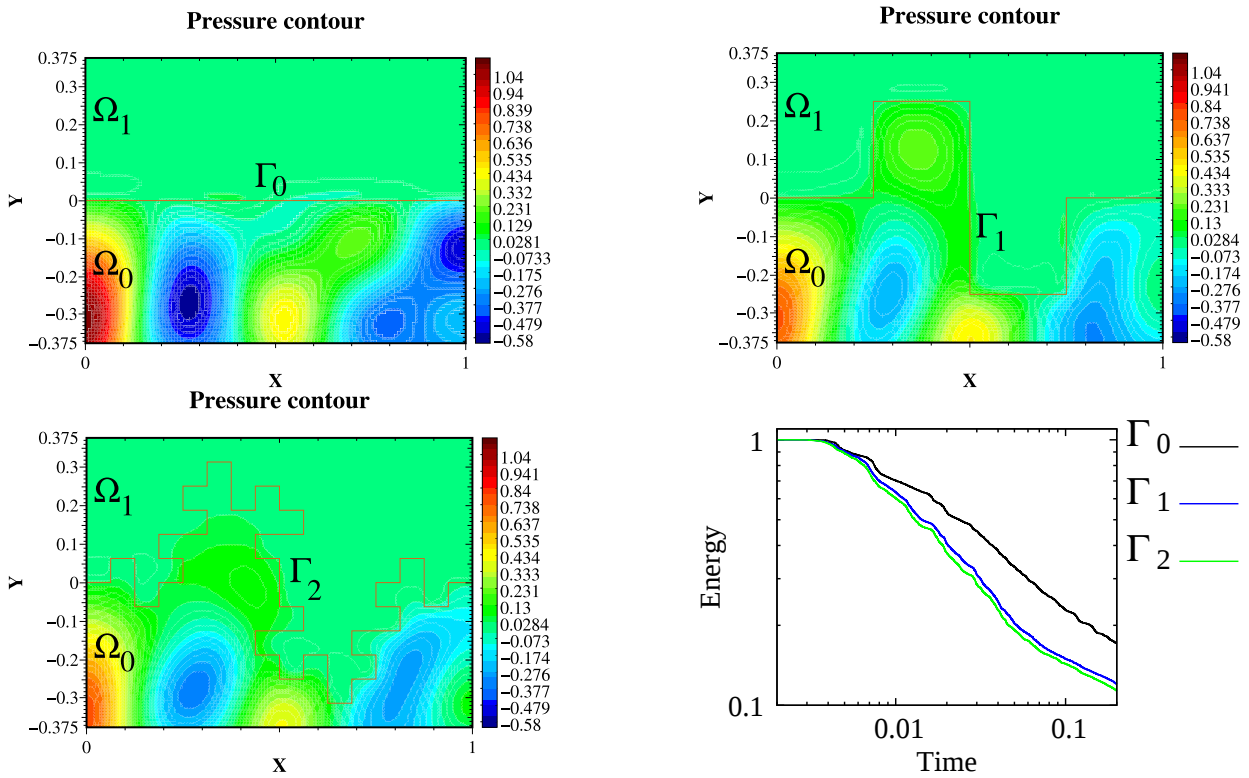


Figure 3.6 – Pressure contours at $t = 0.01$ in cavities with an internal boundary of different Minkowski fractal generations (from left to right and top to bottom: Γ_0 (flat), Γ_1 and Γ_2) and the corresponding energy damping. The size of the mesh is 128×512 .

$$\xi(x)\partial_t^2 u + a(x)\partial_t u - \nabla \cdot (\eta(x)\nabla u) = 0 \quad \text{in } \Omega = \Omega_0 \sqcup \Omega_1, \quad (3.13)$$

with $\xi(x) = \frac{1}{c_0^2}$, $a(x) = 0$, $\eta(x) = 1$ in the air and $\xi(x) = \frac{\phi\gamma_p}{c_0^2\rho_0\alpha_h}$, $a(x) = \sigma\frac{\phi^2\gamma_p}{c_0^2\rho_0\alpha_h}$, $\eta(x) = \frac{\phi}{\alpha_h}$ in the porous medium. Equation (3.13) is supplemented with the following no-jump conditions through Γ_i

$$[u]_{\Gamma_i} = [\eta\nabla u \cdot \nu]_{\Gamma_i} = 0$$

and with an initial data chosen as a Gaussian, centered in a fixed point $x_0 = (0.75, -1.5)$ of Ω_0 :

$$u|_{t=0} = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{|x-x_0|^2}{2\delta^2}}, \quad \partial_t u|_{t=0} = 0,$$

where $\delta = 0.1$. Such a choice of δ ensures that $\text{supp}(u|_{t=0})$ is extremely small outside Ω_0 . Equation (3.13) is the wave equation in the air ($a = 0$), and is the damped wave equation [7, 18] ($a \neq 0$) in the porous medium. This provides the energy decay (see TD1):

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} [\xi(\partial_t u)^2 + (\mu \nabla u) \cdot \nabla u] dx \right) = - \int_{\Omega} a(\partial_t u)^2 dx. \quad (3.14)$$

We discretize Eq. (3.13) in a way which mimics the energy dissipation (3.14), and which is an adaptation to damped acoustic waves of the finite volume method presented in Ref. [42]. Let u_i^n be the discretized pressure in the control volume i at time $n\Delta t$, then we write

$$\xi \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + a \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - [\nabla \cdot (\mu \nabla u^n)]_i = 0,$$

so that the energy like functional

$$E^{n+1/2} := \frac{1}{2} \left(\int_{\Omega} \xi \left(\frac{u^{n+1} - u^n}{\Delta t} \right)^2 dx + \int_{\Omega} \mu \nabla u^n \cdot \nabla u^{n+1} dx \right)$$

is damped as

$$\frac{1}{\Delta t} (E^{n+1/2} - E^{n-1/2}) = - \int_{\Omega_1} a \left(\frac{u^{n+1} - u^{n-1}}{2\Delta t} \right)^2 dx.$$

Fig. 3.6 shows that an irregular shape of the internal boundary can significantly increase the dissipation properties of the porous medium ($\Gamma_{1,2}$ as compared to Γ_0). The energy damping by Γ_1 , compared to the damping performances of Γ_0 , is much better, and we notice that the wavelength λ of the wave, created by the initial data, is comparable (twice bigger) to the characteristic length scale size of the geometry Γ_1 . At the same time, the small difference in the energy decays corresponding to the internal boundaries Γ_1 and Γ_2 confirms that the optimal interaction of the wave with the wall happens for the geometric fragments of the size $\lambda/2$. We see that the wave does not penetrate in the smallest geometry parts of size $\lambda/8$, but the wave still keeps a good penetration for the scales of the order $\lambda/2$ as for Γ_1 . This finally implies that the shape of the internal boundary does not need to be “too complicated” for being an efficient acoustic absorbent for a fixed frequency.

3.2.3 Damping by the boundary

Instead of energy absorption in volume, we can also consider the following frequency model of damping by the boundary. Let Ω be a connected bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$.

We suppose that the boundary $\partial\Omega$ is divided into three parts $\partial\Omega = \Gamma_{Dir} \cup \Gamma_{Neu} \cup \Gamma$ (see Fig. 3.7 for an example of Ω) and consider

$$\begin{cases} \Delta u + \omega^2 u = f(x), & x \in \Omega, \\ u = g(x) & \text{on } \Gamma_{Dir}, \quad \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_{Neu}, \quad \frac{\partial u}{\partial n} + \alpha(x)u = \text{Tr}h(x) & \text{on } \Gamma, \end{cases} \quad (3.15)$$

where $\alpha(x)$ is a complex-valued regular function with a strictly positive real part ($\text{Re}(\alpha) > 0$) and a strictly negative imaginary part ($\text{Im}(\alpha) < 0$).

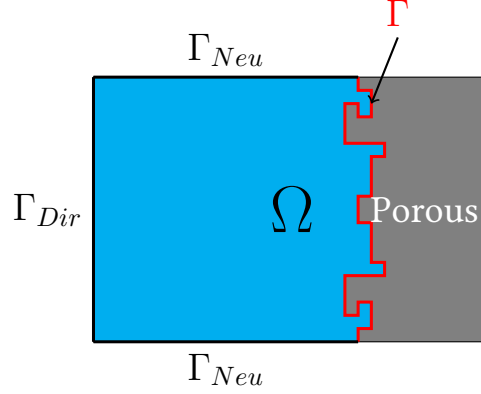


Figure 3.7 – Example of a domain Ω in $\mathbb{R}^2$, shown in blue, with three types of boundaries: Γ_{Dir} , where there is a source of the noise and Γ_{Neu} for the reflective part (for instance of a room or of a tunnel) and Γ on which we impose an absorbing boundary condition giving an equivalent energy decay as for the model with damping in the volume.

Remark: This particular choice of the signs of the real and the imaginary parts of α are needed for the well-posedness properties [30] and the energy decay of the corresponding time-dependent problem. In addition, as the frequency $\omega > 0$ is supposed to be fixed, α can contain a dependence on ω , *i.e.*, $\alpha \equiv \alpha(x, \omega)$.

Problem (5.7) is a frequency version of the following time-dependent wave propagation problem with $U(t, x) = e^{-i\omega t}u(x)$, considered in Ref. [10] for $g = 0$ on Γ_{Dir} :

$$\partial_t^2 U - \Delta U = -e^{-i\omega t} f(x), \quad (3.16)$$

$$U|_{t=0} = U_0, \quad \partial_t U|_{t=0} = U_1, \quad (3.17)$$

$$U|_{\Gamma_{Dir}} = g, \quad \left. \frac{\partial U}{\partial n} \right|_{\Gamma_{Neu}} = 0, \quad (3.18)$$

$$\frac{\partial U}{\partial n} - \frac{\text{Im}(\alpha(x))}{\omega} \partial_t U + \text{Re}(\alpha(x)) U|_{\Gamma} = 0. \quad (3.19)$$

To show the energy decay, we follow [10] and introduce the Hilbert space $X_0(\Omega)$, defined as the Cartesian product of the set of functions $u \in H^1(\Omega)$, which vanish on Γ_{Dir} with the space $L^2(\Omega)$. The equivalent norm on $X_0(\Omega)$ is defined by

$$\|(u, v)\|_{X_0(\Omega, \mu)}^2 = \int_{\Omega} (|\nabla_x u|^2 + |v|^2) dx + \int_{\Gamma} \text{Re}(\alpha(x)) |\text{Tr} u|^2 d\mu$$

with the corresponding inner product

$$\langle (u_1, u_2), (v_1, v_2) \rangle = \int_{\Omega} (\nabla_x u_1 \nabla_x v_1 + u_2 v_2) dx + \int_{\Gamma} \text{Re}(\alpha(x)) \text{Tr} u_1 \text{Tr} v_1 d\mu. \quad (3.20)$$

Here μ is a finite positive d -measure on Γ ($n-1 \leq d < n$), which in the case of a regular Γ (at least Lipschitz), if there are no specific assumptions, is equal to the Lebesgue measure on Γ , and in this case is denoted by λ .

The advantage of this norm is that the energy balance of the homogeneous problem (3.16)–(3.19) has the form (see TD 1)

$$\partial_t \left(\|(U, \partial_t U)\|_{X_0(\Omega, \mu)}^2 \right) = \frac{2}{\omega} \int_{\Gamma} \operatorname{Im}(\alpha(x)) |\operatorname{Tr} \partial_t U|^2 d\mu.$$

Therefore, for $\operatorname{Im}(\alpha) < 0$ on Γ , the energy decays in time. We also notice that the positive real part of α add a positive term in the definition of the energy and thus describe the reflection of Γ . For the case of a smooth boundary $\partial\Omega$ (at least Lipschitz), we have the well-posedness of both models. Thanks to [10], for all $f \in L^2(\Omega)$, $(U_0, U_1) \in X_0(\Omega)$ there exists a unique solution $(U, U_t) \in C([0, \infty[, X_0(\Omega))$ of system (3.16)–(3.19) under the assumption that $\operatorname{Re}(\alpha(x)) > 0$ and $\operatorname{Im}(\alpha(x)) < 0$ are continuous functions.

But it is also possible to have the well posedness on domains with a d -set self-similar fractal boundaries $n - 1 \leq d < n$. (See TD3 or for more details [53]). More precisely, for $g = 0$ we have (the result will be partially proved in TDs)

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a (compact) Lipschitz or self-similar d -set boundary $\partial\Omega$ ($n - 1 < d < n$) with a d -measure μ . Assume $\partial\Omega = \Gamma_{Dir} \cup \Gamma_{Neu} \cup \Gamma$ such that it holds (1.25), $\mu(\Gamma) > 0$, $\Gamma \subseteq \partial\Omega$ and Γ_{Dir} and Γ are also compact with the same properties as $\partial\Omega$ itself. Let in addition $g = 0$ and $\operatorname{Re}(\alpha) > 0$, $\operatorname{Im}(\alpha) < 0$ on Γ (α is a continuous function or simply a constant).*

Then for all $f \in L^2(\Omega)$, $h \in L^2(\Gamma, \mu)$, and $\omega > 0$ (or equivalently, $k > 0$) there exists a unique solution $u \in V(\Omega)$ (the space $V(\Omega)$ is defined in (1.27)) of the Helmholtz problem (3.15) in the following sense: for all $v \in V(\Omega)$ it holds (see also (7.8))

$$\int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \omega^2 \int_{\Omega} u \bar{v} dx + \int_{\Gamma} \alpha \operatorname{Tr} u \operatorname{Tr} \bar{v} d\mu = - \int_{\Omega} f \bar{v} dx + \int_{\Gamma} \operatorname{Tr} h \operatorname{Tr} \bar{v} d\mu. \quad (3.21)$$

Moreover, the solution of problem (3.21) $u \in V(\Omega)$, continuously depends on the data: there exists a constant $C > 0$, depending only on $\|\operatorname{Tr}\|_{\mathcal{L}(V(\Omega), L^2(\Gamma, \mu))}$, $C_P(\Omega)$ (the Poincaré constant associated to Ω), α and ω , such that

$$\|u\|_{V(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\Gamma, \mu)} \right). \quad (3.22)$$

In addition, if, for $m \in \mathbb{N}^$, $\partial\Omega \in C^{m+2}$ (hence μ is an $(n - 1)$ -measure on $\partial\Omega$), $f \in H^m(\Omega)$, $h = 0$, then the solution u belongs to $H^{m+2}(\Omega)$.*

For the treatment of the non homogeneous Dirichlet boundary condition see Remark 7.4.3.

Remark: Going back to the situation of Fig. 3.1 we could imagine the following configuration presented on Fig. 3.8.

3.2.4 Calculation of the complex coefficient α in the Robin boundary condition for a fixed porous material

We know the macroscopic physical characteristics of a fixed porous medium as porosity ϕ , tortuosity α_h and resistivity to the passage of air σ . We then know how to define the corresponding

$$\begin{array}{c}
 \frac{\partial u}{\partial n} - iku|_{\Gamma_{abs}} = 0 \\
 \\
 u|_{\Gamma_{Dir}} = g(y) \quad \Delta u + k^2 u = 0 \text{ on } \Omega, k = \frac{\omega}{c} > 0 \quad \frac{\partial u}{\partial n} + \alpha(\omega)u|_{\Gamma} = 0 \\
 \\
 \frac{\partial u}{\partial n}|_{\Gamma_{Neu}} = 0
 \end{array}$$

Figure 3.8 – Typical model for the situation of Fig. 3.1.

energy decay from the damped wave equation. But mostly for the numerical simplifications, we would like to work with a model of the wave propagation in the air with a boundary giving us the shape of an absorbing wall and on which we impose a Robin type complex boundary condition

$$\frac{\partial u}{\partial n} + \alpha(\omega)u|_{\Gamma} = 0,$$

ensuring the same wave propagation behavior as for the case of damping in the volume (see Fig. 3.9).

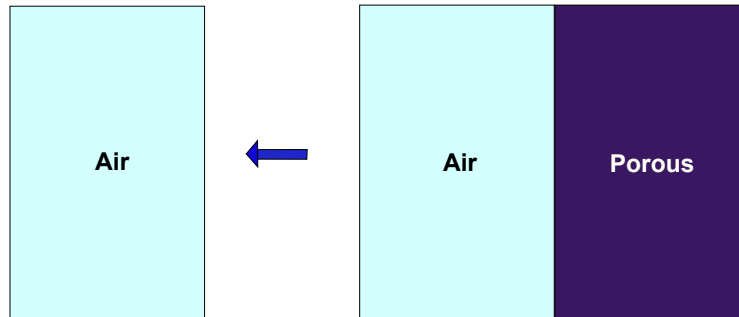


Figure 3.9 – We want to find a condition on the boundary of the wall which ensures that the solution of the one medium problem is almost the same as the solution in the air of the two media problem.

In other words we need to find the coefficient α from the following minimization problem, once again for the simplicity considered for a plane shape of the wall:

$$\left\{ \begin{array}{l} \text{in } \Omega_{air} : \\ \nabla \cdot (\eta_0 \nabla u_0) + \omega^2 \xi_0 u_0 = 0 \\ \text{in } \Omega_{wall} : \\ \nabla \cdot (\eta_1 \nabla u_1) + \omega^2 \xi_1 \left(1 + \frac{ai}{\xi_1 \omega}\right) u_1 = 0 \\ \text{on } \Gamma : \\ u_0 = u_1 \text{ and } \eta_0 \nabla u_0 \cdot \nu = \eta_1 \nabla u_1 \cdot \nu \\ \text{on the left boundary:} \\ u_0(-L, y) = g(y) \end{array} \right\} \quad \left\{ \begin{array}{l} \text{in } \Omega_{air} : \\ \nabla \cdot (\eta_0 \nabla u_2) + \omega^2 \xi_0 u_2 = 0 \\ \text{on } \Gamma : \\ \eta_0 \nabla u_2 \cdot \nu + \alpha u_2 = 0 \\ \text{on the left boundary:} \\ u_2(-L, y) = g(y) \end{array} \right.$$

Find α such that for constants $A \geq 0$ and $B \geq 0$

$$A\|u_0 - u_2\|_{L^2(\Omega_{air})}^2 + B\|\nabla(u_0 - u_2)\|_{L^2(\Omega_{air})}^2 \rightarrow \min.$$

Thus it is possible to solve this problem numerically, using the formulas of the following Theorem (the proof is given in [53], we don't ask for it in this course):

Theorem 3.2.2. *Let $\Omega =]-L, L[\times]-\ell, \ell[$ be a domain with a simply connected sub-domain Ω_{air} , whose boundaries are $]-L, 0[\times \{\ell\}$, $\{-L\} \times]-\ell, \ell[$, $]-L, 0[\times \{-\ell\}$ and another boundary, denoted by Γ , which is the straight line starting in $(0, -\ell)$ and ending in $(0, \ell)$. In addition let Ω_{wall} be the supplementary domain of Ω_{air} in Ω , so that Γ is the common boundary of Ω_{air} and Ω_{wall} . The length L is supposed to be large enough.*

Let the original problem (the frequency version of the wave damped problem (3.12)) be

$$-\nabla \cdot (\eta_0 \nabla u_0) - \omega^2 \xi_0 u_0 = 0 \quad \text{in } \Omega_{air}, \quad (3.23)$$

$$-\nabla \cdot (\eta_1 \nabla u_1) - \omega^2 \tilde{\xi}_1 u_1 = 0 \quad \text{in } \Omega_{wall}, \quad (3.24)$$

with

$$\tilde{\xi}_1 = \xi_1 \left(1 + \frac{ai}{\xi_1 \omega} \right),$$

together with boundary conditions on Γ

$$u_0 = u_1 \quad \text{and} \quad \eta_0 \nabla u_0 \cdot \nu = \eta_1 \nabla u_1 \cdot \nu, \quad (3.25)$$

and the condition on the left boundary

$$u_0(-L, y) = g(y), \quad (3.26)$$

and some other boundary conditions. Let the modified problem be

$$-\nabla \cdot (\eta_0 \nabla u_2) - \omega^2 \xi_0 u_2 = 0 \quad \text{in } \Omega_{air} \quad (3.27)$$

with boundary absorption condition on Γ

$$\eta_0 \nabla u_2 \cdot \nu + \alpha u_2 = 0 \quad (3.28)$$

and the condition on the left boundary

$$u_2(-L, y) = g(y). \quad (3.29)$$

Let u_0, u_1, u_2 and g be decomposed into Fourier modes in the y direction, denoting by k the associated wave number. Then the complex parameter α , minimizing the following expression

$$A\|u_0 - u_2\|_{L^2(\Omega_{air})}^2 + B\|\nabla(u_0 - u_2)\|_{L^2(\Omega_{air})}^2$$

can be found from the minimization of the error function

$$e(\alpha) := \sum_{k=\frac{n\pi}{\ell}, n \in \mathbb{Z}} e_k(\alpha),$$

where e_k are given by

$$e_k(\alpha) = (A + B|k|^2) \left(\frac{1}{2\lambda_0} \left\{ |\chi|^2 [1 - \exp(-2\lambda_0 L)] + |\gamma|^2 [\exp(2\lambda_0 L) - 1] \right\} + 2L \operatorname{Re}(\chi \bar{\gamma}) \right) + B \frac{\lambda_0}{2} \left\{ |\chi|^2 [1 - \exp(-2\lambda_0 L)] + |\gamma|^2 [\exp(2\lambda_0 L) - 1] \right\} - 2B\lambda_0^2 L \operatorname{Re}(\chi \bar{\gamma})$$

if $k^2 \geq \frac{\xi_0}{\eta_0} \omega^2$ or

$$e_k(\alpha) = (A + B|k|^2) \left(L(|\chi|^2 + |\gamma|^2) + \frac{i}{\lambda_0} \operatorname{Im} \{ \chi \bar{\gamma} [1 - \exp(-2\lambda_0 L)] \} \right) + BL|\lambda_0|^2 (|\chi|^2 + |\gamma|^2) + iB\lambda_0 \operatorname{Im} \{ \chi \bar{\gamma} [1 - \exp(-2\lambda_0 L)] \}$$

if $k^2 < \frac{\xi_0}{\eta_0} \omega^2$, in which

$$\begin{aligned} f(x) &= (\lambda_0 \eta_0 - x) \exp(-\lambda_0 L) + (\lambda_0 \eta_0 + x) \exp(\lambda_0 L), \\ \chi(k, \alpha) &= g_k \left(\frac{\lambda_0 \eta_0 - \lambda_1 \eta_1}{f(\lambda_1 \eta_1)} - \frac{\lambda_0 \eta_0 - \alpha}{f(\alpha)} \right), \\ \gamma(k, \alpha) &= g_k \left(\frac{\lambda_0 \eta_0 + \lambda_1 \eta_1}{f(\lambda_1 \eta_1)} - \frac{\lambda_0 \eta_0 + \alpha}{f(\alpha)} \right), \end{aligned}$$

where

$$\begin{cases} \lambda_0 = \sqrt{k^2 - \frac{\xi_0}{\eta_0} \omega^2} & \text{if } k^2 \geq \frac{\xi_0}{\eta_0} \omega^2, \\ \lambda_0 = i \sqrt{\frac{\xi_0}{\eta_0} \omega^2 - k^2} & \text{if } k^2 \leq \frac{\xi_0}{\eta_0} \omega^2. \end{cases} \quad (3.30)$$

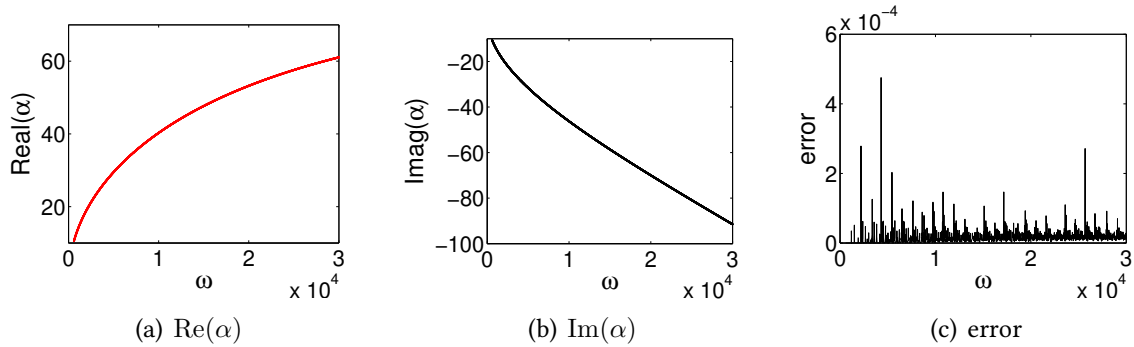


Figure 3.10 – The real (top left) and imaginary (top right) parts of α and the sum of the errors $e_{\Delta x}$ (in the bottom) as function of frequencies $\omega \in [600, 30000]$ calculated for the ISOREL porous material.

Since the minimization will be done numerically and since the sequence $(z, -z, z - z, \dots) = z(\exp(i(j\Delta x)/\Delta x))$ is the highest frequency mode that can be reached on a grid of size Δx , then, in practice, the sum may be truncated to

$$e_{\Delta x}(\alpha) := \sum_{k = \frac{n\pi}{L}, n \in \mathbb{Z}, -\frac{L}{\Delta x} \leq n \leq \frac{L}{\Delta x}} e_k(\alpha).$$

For the equations (3.23)– (3.24), we use the same coefficients as for problem (3.12) and take the values corresponding to a porous medium, called ISOREL, using in building insulation. More

precisely we assume: $\phi = 0.7$, $\gamma_p = 1.4$, $\sigma = 142300 N.m^{-4}.s$, $\rho_0 = 1.2 kg/m^3$, $\alpha_h = 1.15$, $c_0 = 340 m.s^{-1}$. We could find the value of α presented in Fig. 3.10.

Remark: Fig. 3.10 allows us to compare the difference between two considered time-dependent models for the damping in the volume and for the damping on the boundary. We see that $\text{Re}(\alpha)$ is not a constant in general, but for $\omega \rightarrow +\infty$ $\text{Im}(\alpha)$ is a linear function of ω . In this sense, the damping properties of two models are almost the same, but the reflection is more accurately considered by the damping wave equation in the volume.

Chapter 4

Localization phenomena

The localization of the waves, their energy, the eigenmodes, and their role in the exchange of energy and absorption are fundamental questions in physics and their applications. From a mathematical point of view, most of these questions are not yet really understood and mathematically analyzed. However, there are some partial results. A relatively complete survey about the localization phenomena is given in [35]. For the mathematical explication of the localization for elliptic operators and the Anderson localization in the homogeneous Dirichlet boundary condition framework, see [29].

In this course we will briefly see what it is mostly numerically and physically known for the localization phenomena. In particular, we will discuss how they are related to the shape optimization for more efficient absorption of a wave's energy. Our goal is to understand that we need to create the localization for efficient absorption.

4.1 Spectral problems for $-\Delta$

Let us start by introducing the spectral problems for $-\Delta$, which can be considered with different types of homogeneous boundary conditions, as the Dirichlet, the Neumann, the Robin or mixed type boundary conditions (see Chapter 1).

For instance, let us start with the Dirichlet Laplacian on a domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \quad (4.1)$$

This spectral problem (see Definition 6.4.1) is usually understood in the weak sense:

$$\forall v \in H_0^1(\Omega) \quad \int_{\Omega} \nabla u \cdot \nabla v dx = \lambda \int_{\Omega} uv dx. \quad (4.2)$$

This problem is solved in TD 3 (for the Robin Laplacian see Section 2 of TD 3) using the Hilbert-Schmidt Theorem 6.4.1 for the compact auto-adjoint operators on a Hilbert space. Adding the results of [24] (Section 6.5.1 p. 334) we can summarize the results in the following theorem:

Theorem 4.1.1. *Let Ω be a bounded arbitrary domain in $\mathbb{R}^n$ ($n \geq 2$). Then the spectral problem for $-\Delta$ with homogeneous Dirichlet boundary conditions (4.2) has the following properties:*

1. *The eigenvalues of $-\Delta$ are at most countable, of finite multiplicity and strictly positive.*

2. If we repeat each eigenvalue according to its multiplicity, we have the sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

with

$$\lambda_k \rightarrow +\infty \text{ for } k \rightarrow +\infty.$$

3. The corresponding sequence of eigenfunctions $\{\phi_k\}_{k=1}^\infty$ forms an orthonormal basis of $L^2(\Omega)$, an orthogonal basis of $H_0^1(\Omega)$ and for all $k \in \mathbb{N}$ $\phi_k \in H_0^1(\Omega) \cap C^\infty(\Omega)$.

Remark: The key point to work on arbitrary domain is the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ [24]. In TD3 it was shown that $\{\phi_k\}_{k=1}^\infty$ is an orthogonal basis of $H_0^1(\Omega)$ with $\|\phi_k\|_{L^2(\Omega)} = 1$. Thus, it is sufficient to notice that $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ to conclude that $\{\phi_k\}_{k=1}^\infty$ forms an orthonormal basis of $L^2(\Omega)$.

If we consider (4.2) for all $\phi \in H^1(\Omega)$ we obtain the spectral problem of the Neumann Laplacian. This time the compactness of the embedding $H^1(\Omega)$ into $L^2(\Omega)$ needs some geometrical restrictions on $\partial\Omega$ (typically Ω should be an NTA-domain (see Chapter 1), or more generally an extension domain, but we don't study it in this course [5]). The same assumption on Ω is needed for the Robin or mixed Laplacian. In the case of this “regular” Ω , the spectral problem for the Neumann Laplacian has the same properties as the Dirichlet Laplacian, except that $\lambda_1 = 0$ and the corresponding eigenfunction w_1 is constant on Ω (independently on Ω).

It is important to notice that all eigenfunctions are $C^\infty(\Omega)$ (independently on the cited boundary conditions).

Physicists prefer to call eigenfunctions by **eigenmodes**.

Knowing that it is possible for the Dirichlet Laplacian to solve the eigenvalue problem (4.2) for all arbitrary shapes of $\partial\Omega$, the eigenmodes depend considerably from the shape of the boundary. Actually, the shape of the boundary may or may not provide localization of some eigenmodes.

4.2 Localization phenomena

There is no a strict mathematical definition of the localization of the eigenmodes, but it rather intuitive (see Fig. 4.1), and could be characterized by a simple phrase “Localized eigenfunctions: here you see them, there you don't” [40].

More precisely, we could say that a localization of an eigenfunction ϕ is the existence of a subdomain Ω_{loc} in Ω such that the values of the eigenfunction in the closer of this subdomain $\overline{\Omega}_{loc}$ are “much” or “considerably” larger by modulus than in its complement, *i.e.* in $\overline{\Omega} \setminus \overline{\Omega}_{loc}$. In other words, in $\overline{\Omega} \setminus \overline{\Omega}_{loc}$ the eigenfunction is considerably smaller by modulus than in $\overline{\Omega}_{loc}$:

$$\max_{x \in \overline{\Omega}_{loc}} |\phi(x)| \gg \max_{x \in \overline{\Omega} \setminus \overline{\Omega}_{loc}} |\phi(x)|.$$

In the most cases in addition, it is supposed that the volume of Ω_{loc} is much smaller than the initial volume of Ω .

It is clear that not for all Ω a such subdomain Ω_{loc} exists, as it is known from the explicit calculation of eigenfunctions of $-\Delta$ with the Neumann or the Dirichlet boundary conditions on a

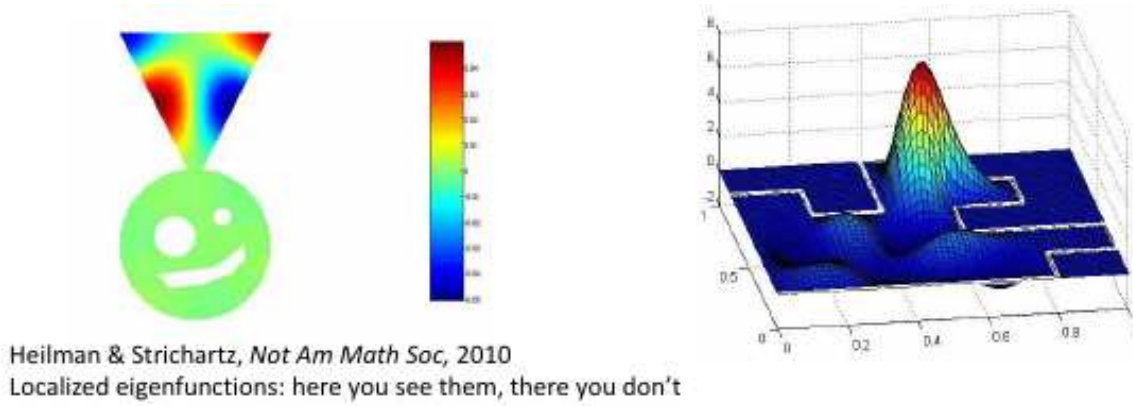


Figure 4.1 – Localization examples by M. Filoche. On the left, it is an eigenfunction of the Neumann Laplacian, and on the right, it is an eigenfunction of the Dirichlet Laplacian.

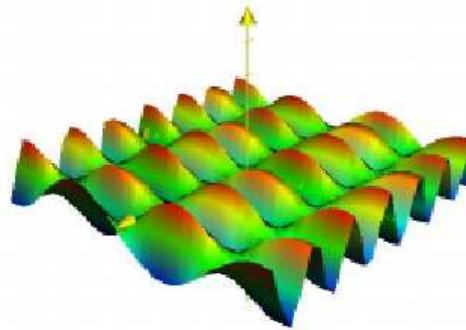


Figure 4.2 – Example of a non localization on a rectangular domain (by M. Filoche).

rectangular domain Ω (see the example of Fig. 4.2). Moreover, in all cases, if Ω allows a localization, not all eigenfunctions are localized: there are two families of eigenfunctions - localized and delocalized, as shown on Figs. 4.5 and 4.6.

In addition, it is not possible to say that $\overline{\Omega}_{loc}$ is the support of a localized eigenfunction ϕ (i.e. that $\overline{\Omega}_{loc} = \overline{\{x \in \Omega | \phi(x) = 0\}}$) since the eigenfunctions cannot be equal to 0 in a subdomain of Ω :

Theorem 4.2.1. *If ϕ is an eigenmode of the Dirichlet (or Neumann, or Robin, or mixed) Laplacian then*

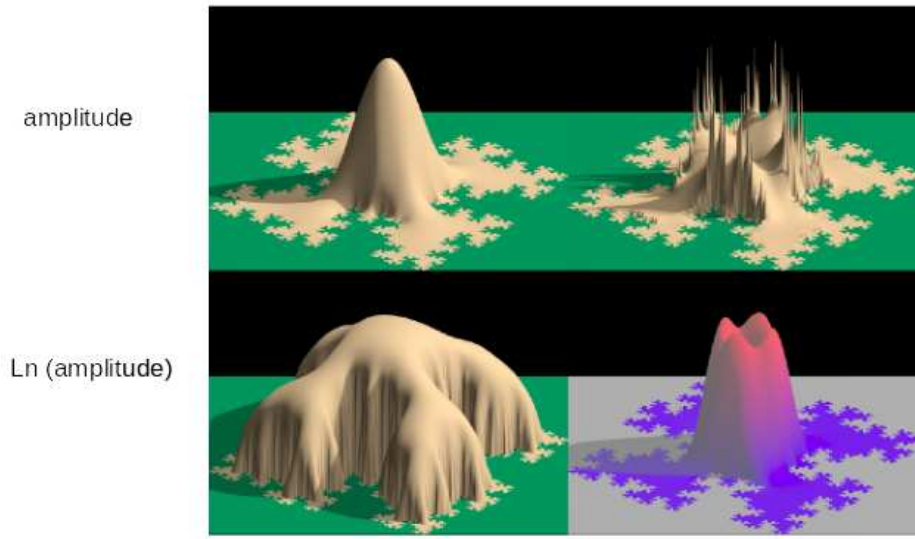
$$\phi \neq 0 \quad \text{a.e. in } \Omega.$$

Actually, if $\phi \equiv 0$ in a subdomain $\Omega_0 \subset \Omega$ then $\phi \equiv 0$ in Ω and hence is not an eigenmode.

Therefore, outside of Ω_{loc} the eigenmode can be small but not identically equal (almost everywhere) to 0. Let ϕ be a localized eigenmode. Typically $\overline{\Omega}_{loc}$ contains the maximum value of $|\phi|$. Thus, passing outside the boundary of Ω_{loc} , there is a decreasing of the modulus of a localized eigenmode.

By the speed of this decreasing of the modulus of a localized eigenfunction, there are notions of

- the strong localization, when the amplitude decrease exponentially or even faster than the exponential decay by works of M. Lapidus (see Fig. 4.3),
- the weak localization, when then the amplitude decrease polynomially (see for example Figs. 4.5, 4.6 and 4.7).



Decay is more rapid than exponential corresponding to strong localization

Figure 4.3 – On the top from left to right: the first eigenfunction of the Dirichlet Laplacian on a prefractal Minkowski domain and the modulus of its gradient. On the bottom, the logarithm of the corresponding amplitudes. It is an example of a strong localization (M. Lapidus).

After it, there is a high-frequency localization and low-frequency localization [35]. The high-frequency localization is more understood from the mathematical point of view. However, it is not interesting for us, as in our framework, the high frequencies are naturally absorbed by the microstructure of a porous material, but what is more important and very difficult to do, is to absorb the low frequencies optimally.

So we consider only the low-frequency localization.

The question of how to characterize a domain, or mainly its boundary, in which there are localized eigenfunctions is mathematically open. From the numerics and experiments (see Fig. 4.4), mainly for the two-dimensional case, we have different examples of domains with some geometrical irregularity as prefractal domains, for which there exist localized eigenmodes in low frequencies. In the Dirichlet boundary case it is more understood [29, 67]

For the Dirichlet boundary case, the subdomains of localization are pouched inside Ω by the constraint to be equal 0 on $\partial\Omega$, and going closer to $\partial\Omega$, there is a decay faster than exponential. In this situation, one of the ideas, why a localization could happen, is related with the idea that it depends on the capacity of the geometry to capture rays (see Chapter 5) or Brownian motion [67]. The localization regions can be defined from the corresponding Green function [29] (we don't go in more details). If this localization region is considered as a separate domain Ω_{loc} with the same type boundary conditions, then the localized in Ω_{loc} eigenfunction for $-\Delta$ on Ω is closed (rather similar) to an eigenfunction found on the domain Ω_{loc} [29] (see Fig. 4.8).

For the Neumann or Robin boundary condition, the maximum of a localized eigenmode is situated on the boundary and the localization subdomain is located near the boundary of Ω (see, for example, Figs. 4.5 and 4.6).

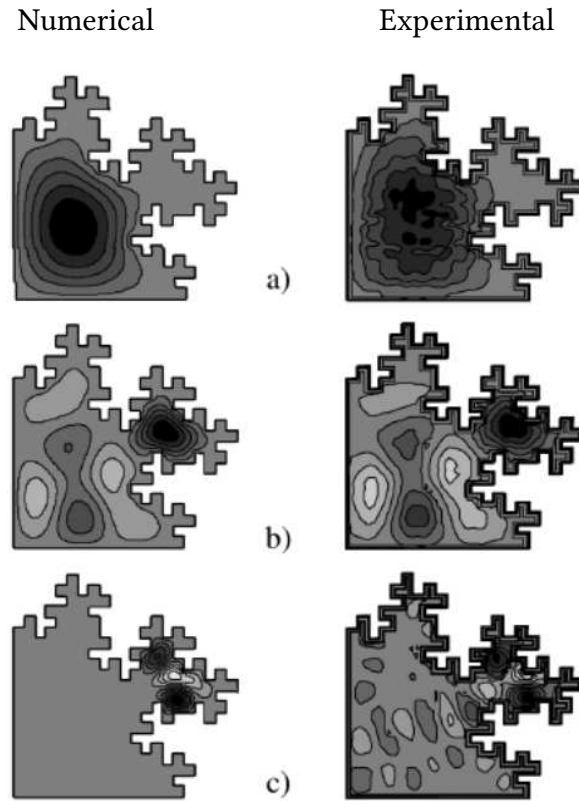


Figure 4.4 – Comparison of the localization of eigenmodes of Dirichlet Laplacian by the numerics and the experience in prefractals [26].

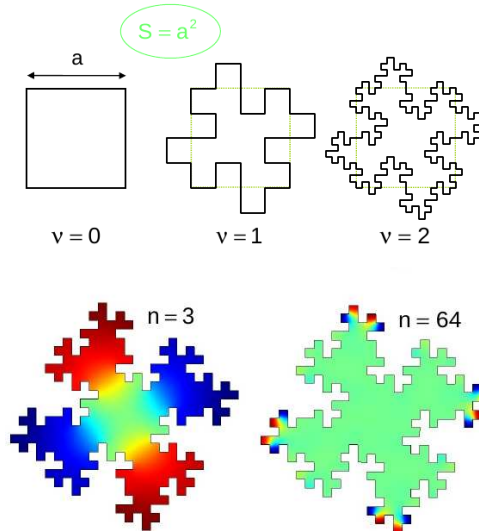


Figure 4.5 – Examples of a delocalized (on the left) and of a localized (on the right) eigenfunctions of the Neumann Laplacian on the second prefractal Minkowski generation [65].

4.3 Localization and increased damping of the waves

4.3.1 Case of the absorption in the volume: astride localization [28]

Astride localization (“à cheval”) Considering the damping in the volume, there are different more or less dissipative models [38]. In Chapter 3, we have discussed the simplest model for

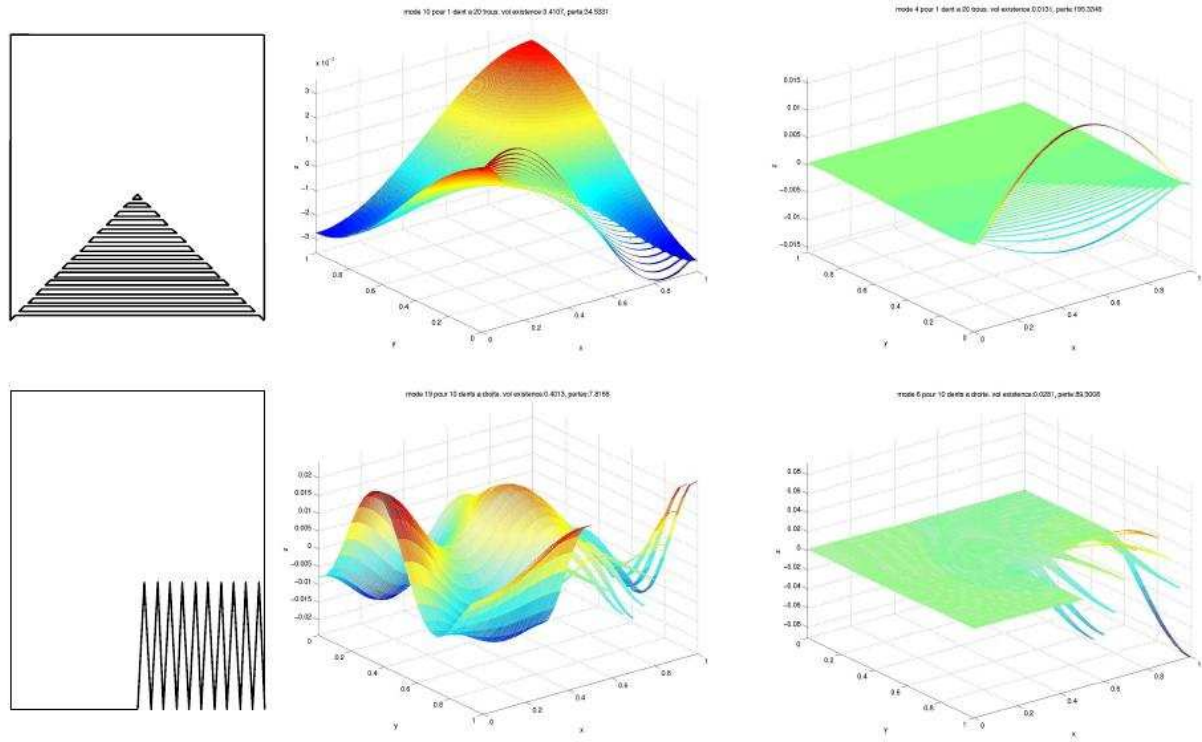


Figure 4.6 – Other examples of a weak localization for the Neumann Laplacian [27]. From the left to the right: the domain geometry, a delocalized eigenmode, a localized eigenmode.

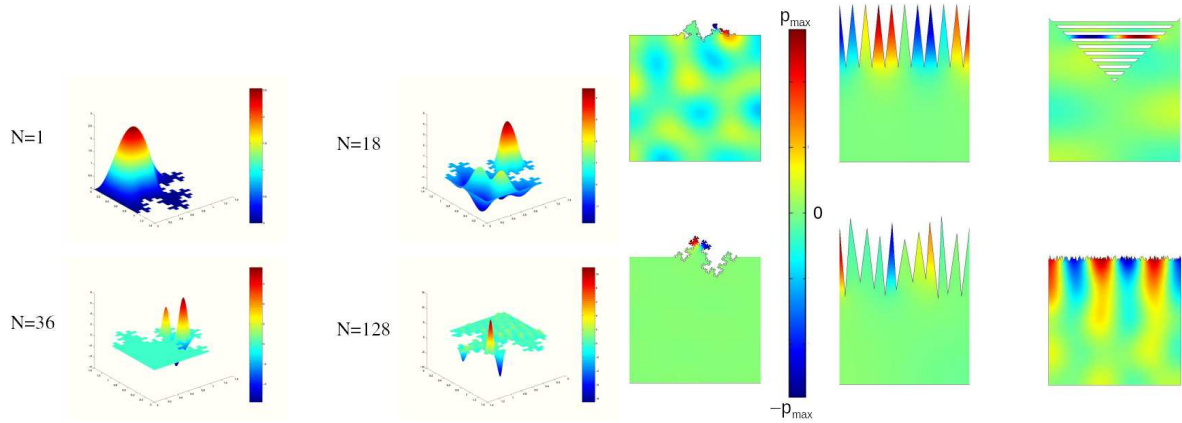


Figure 4.7 – Localization of eigenmodes in prefractals and other type of geometrical irregularities [27]. Two columns on the left (for a Minkowski prefractal part of the boundary), there are examples of eigenfunctions for the Dirichlet Laplacian. Three columns on the right there are examples of eigenfunctions for the Neumann Laplacian.

the propagation in time by the damped wave equation. If we don't consider the propagation in time, but only the frequency models, we can directly consider the following damping eigenvalue problem for $\Omega = \Omega^{air} \sqcup \Omega^{wall}$ (see Fig. 4.9), proposed by J.F. Hamet [38]¹:

¹By its derivation, this model is more dissipative than the frequency variant of the damping wave equation.

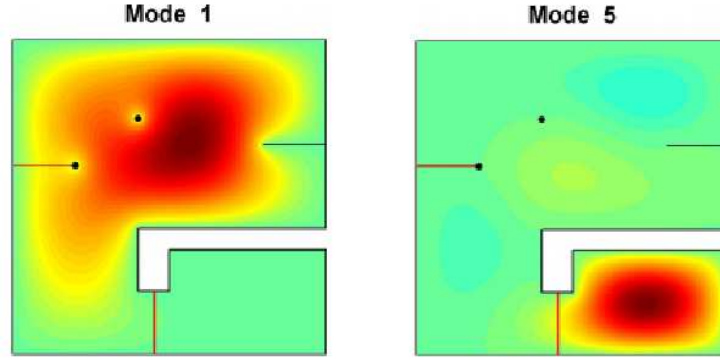


Figure 4.8 – Two predicted area of the localization for the Dirichlet Laplacian. [29] The localized parts are similar to the first eigenmodes on the corresponding subdomains with Dirichlet boundary condition.

$$\begin{aligned}
 -\Delta \phi_k &= \lambda_j \phi_k \quad \text{in } \Omega^{air}, \\
 -\operatorname{div} \left(\frac{1}{\rho_R + i\rho_I} \nabla \phi_k \right) &= \lambda_k \frac{1}{K_R + iK_I} \phi_k \quad \text{in } \Omega^{wall}, \\
 \frac{\partial \phi_k}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \\
 \phi_k|_{\Gamma_-} &= \phi_k|_{\Gamma_+}, \\
 \frac{\partial \phi_k}{\partial \nu} \Big|_{\Gamma_-} &= -\frac{1}{\rho_R + i\rho_I} \frac{\partial \phi_k}{\partial \nu} \Big|_{\Gamma_+}
 \end{aligned}$$

where $\rho_R + i\rho_I$ and $K_R + iK_I$ are effective parameters describing the attenuation of the wave in the air of the porous wall. On the interior boundary Γ between two media we suppose that the eigenfunctions of the problem are continuous and have the gradient jump ensuring that if $\phi_k|_{\Omega^{air}} \in H^1(\Omega^{air})$ and $\phi_k|_{\Omega^{wall}} \in H^1(\Omega^{wall})$ then $\phi_k \in H^1(\Omega)$. The notations Γ_- and Γ_+ show the change of the direction of the external normal on Γ outside of Ω^{air} and Ω^{wall} respectively.

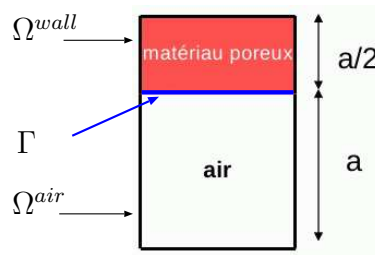


Figure 4.9 – Domain $\Omega = \Omega^{air} \sqcup \Omega^{wall}$ with the plane shape of the wall. The volume of the wall is supposed to be two times smaller than the volume of the air part Ω^{air} [28].

This spectral problem was solved numerically for three shapes of Γ : the plane shape, the first and the second generations of the Minkowski fractal. The wall is supposed to be done with Isorel. As the shapes of the first and the second generations of the Minkowski fractal are symmetrical, then in three considered examples, the volume of the wall (and of the air) is constant. This allows us to compare the absorption properties of these three shapes of the wall. For all considered shapes of Γ there are two families of eigenmodes:

1. a family localized in the air, corresponding to the eigenvalues with the smallest imaginary parts (closed to the real line),
2. a family localized in the wall, corresponding to the eigenvalues with the imaginary parts the most closed to the imaginary parts of the eigenvalues found only for one dissipative medium.

It was also noticed that more the interior boundary Γ becomes irregular (see Figs. 4.10 and 4.11), more there are eigenmodes taking there maximum on Γ , *i.e.* localized astride between the two media. These astride modes correspond to the eigenvalues “in the middle” between the less dissipative and the most dissipative eigenvalues. Actually, the astride localized modes increase the coupling between two media and contribute to a better wave absorption by the geometry [28]. Nevertheless, there are no schematic studies of this type of localization, and there are many related open questions.

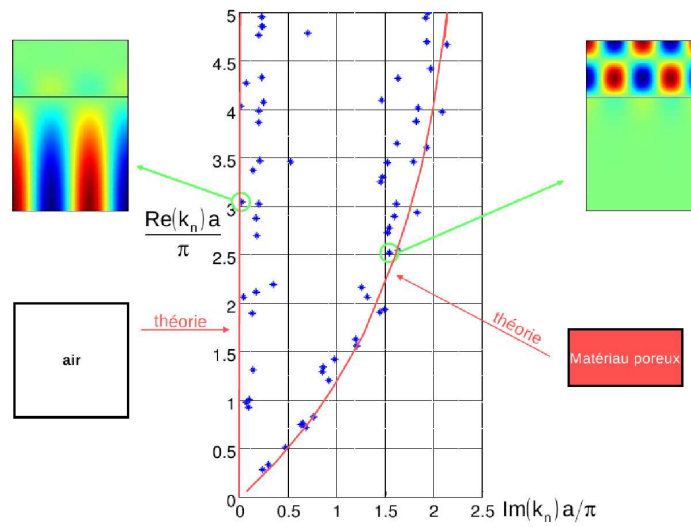


Figure 4.10 – The red lines correspond to the sets containing the eigenfunctions of the spectral problem for only one medium: air (in this case all eigenvalue are strictly positive and real, hence non dissipative) or wall (the non zero imaginary part ensures the dissipative properties). The eigenvalues of the spectral problem for two media are shown by blue stars. We can typically notice that there are two families of eigenfunctions: the family of eigenfunctions corresponding to eigenvalues closed to the real axis, *i.e.* non-dissipative modes localized in the air, and the second family of eigenfunctions corresponding to eigenvalues closed to the dissipative spectral red line, *i.e.* dissipative modes localized in the wall. Here k_n are λ_k in our notations.

4.3.2 Localization of frequency waves on the boundary

Sometimes [27], to simplify the problem, the physicists consider that the homogeneous Neumann boundary condition models the wave’s iteration with a medium with a very small dissipation. For this “pseudo” absorbing boundary, if we consider the solutions of the Helmholtz system with the Neumann boundary condition, we can see [27] on Fig. 4.12 the examples of the localisation on the boundary of the stationary waves.

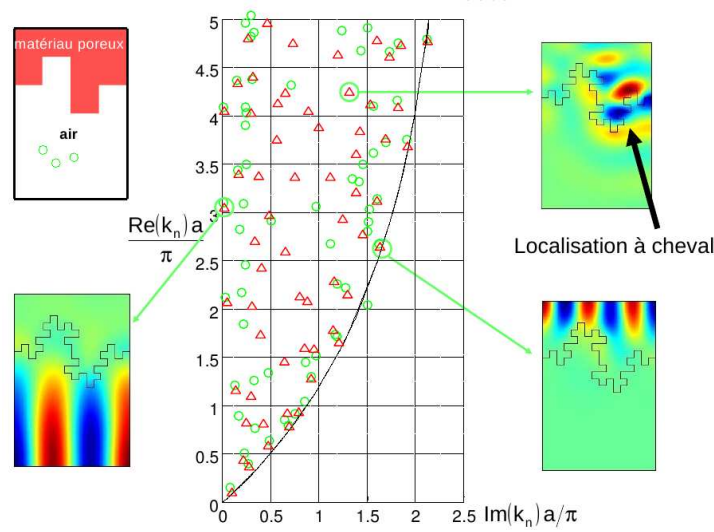


Figure 4.11 – The eigenvalues of two media spectral problem for the first and second generation of the Minkowski fractal denoted by the green circles and red triangles respectively. For the second prefractal shape we distinguish this time three families of eigenfunctions: eigenfunctions, localized in the air, as for the case of the plane shape, with the eigenvalues close to the real line, the second family of eigenfunctions, localized in the wall, also as for the case of the plane shape with the eigenvalues closed to the theoretical line of the eigenvalues for only one dissipative media, but also the third family of eigenfunctions which are localized “astride” (à cheval) between two media, *i.e.* on the interface Γ , and which correspond to the eigenvalues “in the middle” between the eigenvalues of the pure air and pure wall configurations. Here k_n are λ_k in our notations.

4.3.3 Characterization of the localization and of losses

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. To characterize more quantitatively the localization of the eigenfunctions, it was introduced by B. Sapoval and of his co-authors the existence surface of an normalized in $L^2(\Omega)$ eigenmode u_k :

$$\boxed{S_k = \frac{1}{\int_{\Omega} |u_k|^4 dx} \quad \text{for} \quad \int_{\Omega} |u_k|^2 dx = 1.} \quad (4.3)$$

The motivation of this definition and the notion of the existence surface comes from the following remark valid for a constant on Ω function u . Indeed, let

$$u = \begin{cases} F & \text{on } \Omega \\ 0 & \text{on } \mathbb{R}^2 \setminus \Omega \end{cases}$$

Then, with the notation S for the surface area of Ω ,

$$\int_{\Omega} F^2 dx = F^2 S = 1 \quad \Rightarrow \quad F = \frac{1}{S^{\frac{1}{2}}}$$

$$\Rightarrow \quad \frac{1}{\int_{\Omega} |u|^4 dx dy} = \frac{1}{F^4 S} = \frac{1}{\frac{1}{S^2} S} = S,$$

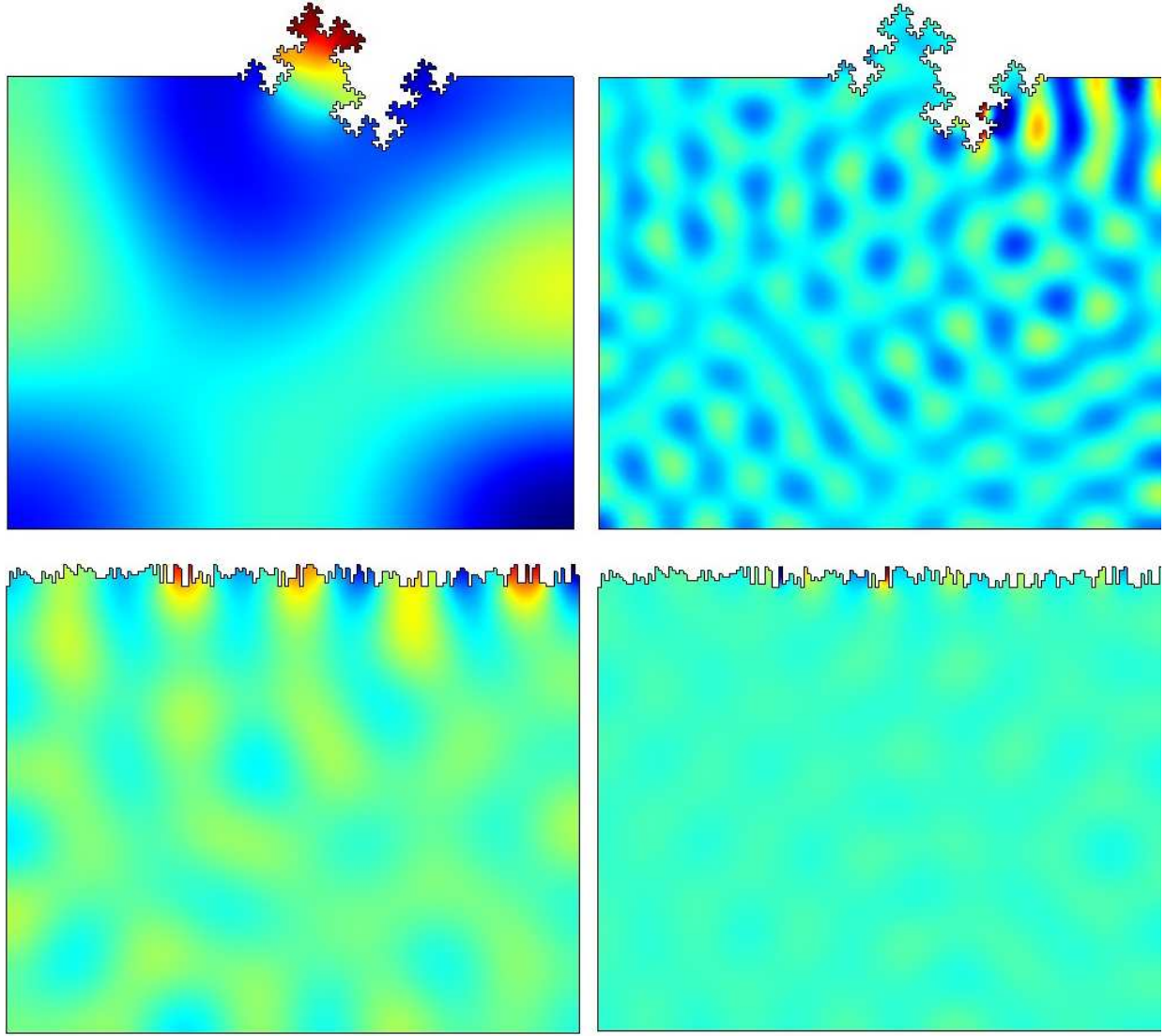


Figure 4.12 – Localization examples of the stationary waves on the boundary modeled by the homogeneous Neumann boundary condition [27].

where we recognize formula (4.3). Therefore, if we normalize the existence surface of an eigenmode number k by the total surface of Ω , we obtain a value $\frac{S_k}{S}$ between 0 and 1. Actually, $\frac{S_k}{S} = 1$ for a constant mode (“living” on all Ω) and we obtain small values near zero for a localized mode in a small are of Ω . Clearly, smaller the value of $\frac{S_k}{S}$, more the eigenmode number k is localized. On Fig. 4.13, we see that the localization phenomena become stronger and stronger for more and more irregular shapes:

Now, let us suppose that a part of the boundary Γ provides an absorption (in [27] it is modeled with the homogeneous Neumann boundary condition) and that Γ has the perimeter equal to L .

Then the energy losses of a L^2 -normalized mode ϕ_k is defined [27] by

$$w_k = \int_{\Gamma} |\phi_k|^2 d\sigma.$$

As the fundamental mode (the first eigenfunction of $-\Delta$ with the Neumann boundary condition corresponding to the eigenvalue $\lambda_0 = 0$) $\phi_0 = \frac{1}{\sqrt{S}}$ (hence, $S_0 = S$), then by the analogy, the

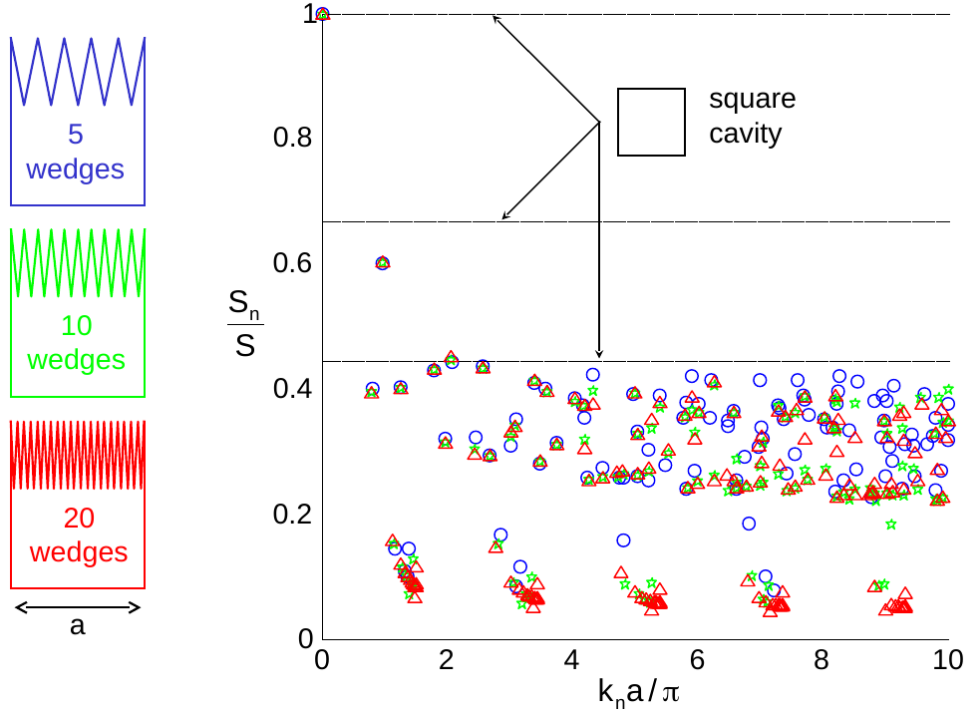


Figure 4.13 – Example of the influence of the irregularity of the Neumann boundary on the localization of eigenmodes of $-\Delta$ [27]. On the abscissa axis, there are (normalized) eigenvalues (here k_n are λ_k in our notations), and on the ordinate axis, the values of the normalized surface existence for each mode (here S_n corresponds to S_k in our notations) are calculated for three given geometries.

eigenmode ϕ_k is approximated by $\frac{1}{\sqrt{S_k}}$. Thus,

$$w_k = \int_{\Gamma} |\phi_k|^2 d\sigma \approx \frac{L_k}{S_k},$$

where L_k is the perimeter of a part of boundary on which the mode ϕ_k is localized ($L_k \leq L$). Then we can directly see with the help of Fig. 4.14, that a localized modes exhibit larger losses: $\frac{L}{S} \ll \frac{L_k}{S_k}$. Besides, Fig. 4.15 shows that the increasing irregularity of a boundary, we increase the losses of the eigenmodes.

Moreover, the losses of the eigenmodes are directly related to their localization: the more the mode is localized, the more important it provides losses (see Fig. 4.16).

4.3.4 Remarks on the wave propagation for the Helmholtz mixed boundary valued problem with a complex Robin boundary condition

Let us go back to our Helmholtz problem with mixed boundary conditions and try to understand what happens with the localization in this case and how it is related to wave absorption.

So, let Ω be a bounded domain with a Lipschitz or a self-similar fractal d -set boundary in $\mathbb{R}^n$. Let in addition, $\partial\Omega = \Gamma_{Dir} \cup \Gamma_{Neu} \cup \Gamma_{Rob}$ as supposed in subsection 1.4.4.

We consider the following eigenvalue problem for $\Delta + \omega^2$ with the mixed boundary conditions,

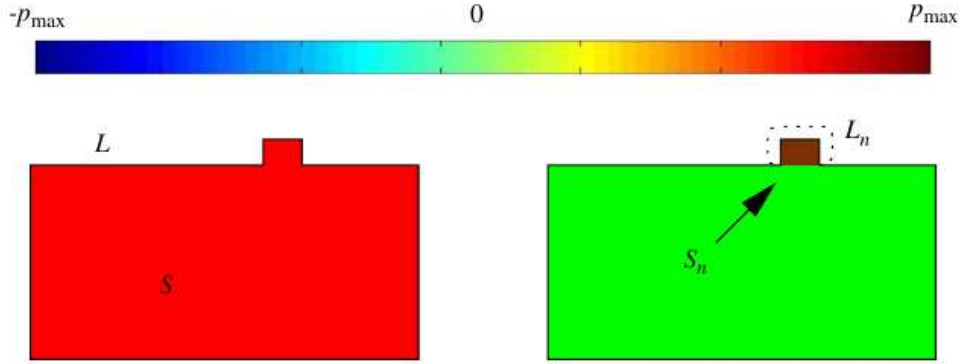


Figure 4.14 – Localized modes exhibit larger losses: $\frac{L}{S} < \frac{L_k}{S_k}$ [27]. For this example we could suppose for the simplicity the domain is the union of a big and of a small square, *i.e.* that $L = 4a + 2b$, $S = a^2 + b^2$, $L_k = 3b$ and $S_k = b^2$. From where, for instance with $a = 1$ and $b < \frac{1}{4}a = \frac{1}{4}$, we directly obtain that $\frac{L}{S} \ll \frac{L_k}{S_k}$.

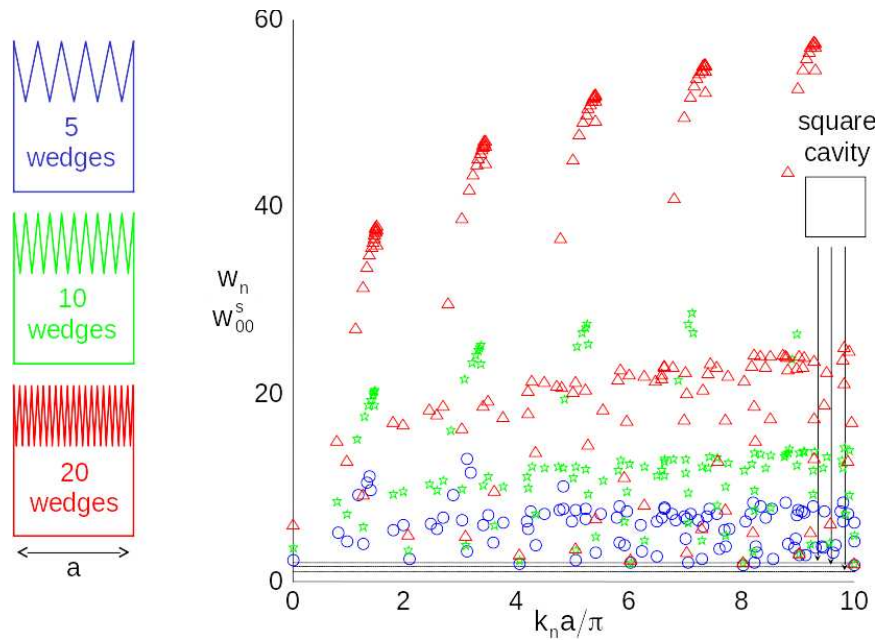


Figure 4.15 – Let for the square cavity $w_{00}^s = \frac{L}{S} = \frac{4}{a}$. The values $\frac{1}{w_{00}^s} \int_{\Gamma} |\phi_k|^2 d\sigma$ as the functions of the eigenvalues of different more and more irregular domains [27]. In our notations: k_n are λ_k .

$\omega > 0$ et $\alpha \in \mathbb{C}$ with $\text{Re } \alpha > 0$ and $\text{Im } \alpha < 0$:

$$\begin{cases} \Delta u + \omega^2 u = \lambda u \text{ dans } \Omega, \\ u = 0 \text{ sur } \Gamma_{Dir}, \\ \frac{\partial u}{\partial n} = 0 \text{ sur } \Gamma_{Neu}, \\ \frac{\partial u}{\partial n} + \alpha u = 0 \text{ sur } \Gamma_{Rob}. \end{cases} \quad (4.4)$$

As α is complex, the solutions (λ, u) are also complex valued.

As in subsection 1.4.4 we define the Hilbert space $V(\Omega)$ by (1.27) with norm (1.28) and the corresponding inner product (1.29).

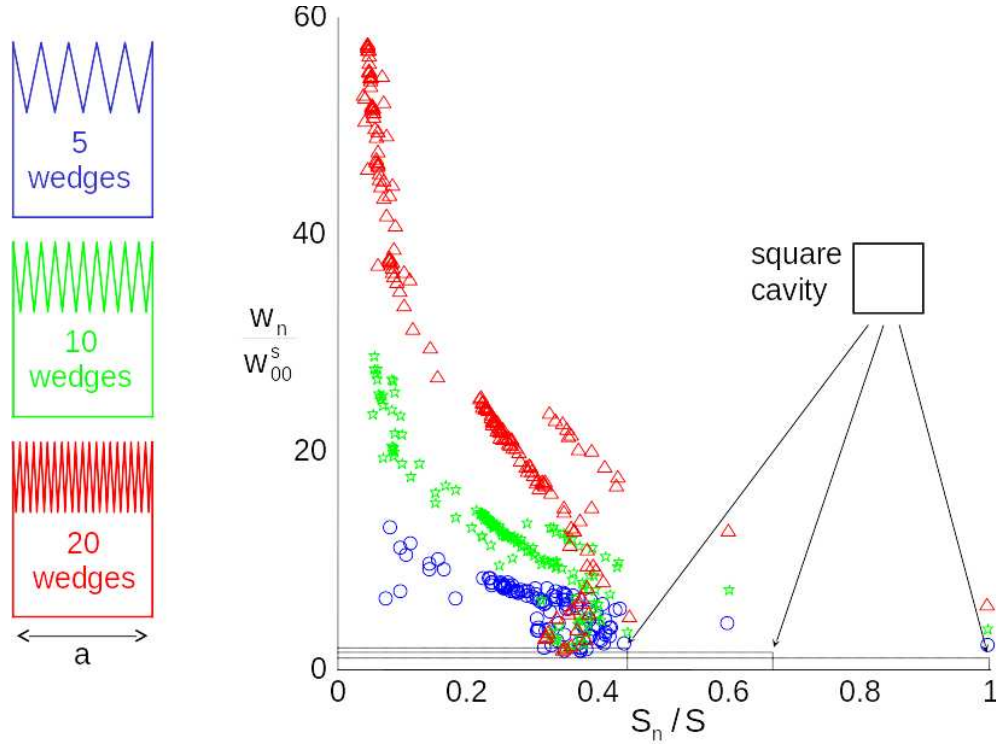


Figure 4.16 – Losses of the modes as the function of their surface existence: more irregular boundaries provide more localization, and a more strong localization provides more energy losses [27].

Definition 4.3.1. We say that $(\lambda, \phi) \in \mathbb{C} \times V(\Omega)$ is a weak solution of problem (4.4) if $\phi \neq 0$ on Ω , $\|\phi\|_{L^2(\Omega)} = 1$ and

$$\forall v \in V(\Omega) \quad - \int_{\Omega} \nabla \phi \cdot \nabla \bar{v} dx + \omega^2 \int_{\Omega} \phi \bar{v} dx - \int_{\Gamma} \alpha \operatorname{Tr} \phi \operatorname{Tr} \bar{v} d\mu = \int_{\Omega} \lambda \phi \bar{v} dx. \quad (4.5)$$

Definition 4.3.2. We say that $(\bar{\lambda}, \psi) \in \mathbb{C} \times V(\Omega)$ is a weak solution of the adjoint problem to (4.4) if $\psi \neq 0$ on Ω , $\|\psi\|_{L^2(\Omega)} = 1$ and

$$\forall v \in V(\Omega) \quad - \int_{\Omega} \nabla \psi \cdot \nabla \bar{v} dx + \omega^2 \int_{\Omega} \psi \bar{v} dx - \int_{\Gamma} \bar{\alpha} \operatorname{Tr} \psi \operatorname{Tr} \bar{v} d\mu = \int_{\Omega} \bar{\lambda} \psi \bar{v} dx. \quad (4.6)$$

It is known that

Theorem 4.3.1. The spectral problem (4.5) admits a countable number of eigenvalues with $\operatorname{Im} \lambda_k \neq 0$ ($\mathbb{R}$ does not belong to the spectrum of Δ , or equivalently, there are no real eigenvalues of Δ with the mixed complex conditions) with $\operatorname{Re} \lambda_k < 0$. If $(\phi_k)_{k \in \mathbb{N}^*}$ are eigenfunctions associated to the eigenvalues $(\lambda_k)_{k \in \mathbb{N}^*}$ and $(\psi_k)_{k \in \mathbb{N}^*}$ are eigenfunctions associated to the eigenvalues $(\bar{\lambda}_k)_{k \in \mathbb{N}^*}$ of the adjoint problem (4.6), then the sequence $(\phi_k, \psi_k)_{k \in \mathbb{N}^*}$ forms an orthonormal basis of $L^2(\Omega)$: $(\phi_j, \psi_k)_{L^2(\Omega)} = \delta_{jk}$.

If $\operatorname{Im} \alpha = 0$, then the eigenvalues of Δ are all strictly negative (therefore real) and in this case there is no energy absorption. So the real part of λ_k influences the reflection.

At the same time, more larger is the value of $|\operatorname{Im} \lambda_k|$, more dissipative the eigenfunction is. So we are interested in fixing ω and in finding the least dissipative eigenmodes (functions) and the most dissipative eigenfunctions.

Let us give a remark on the classical decomposition of a solution of a non dissipative wave equation over the eigenfunctions of $-\Delta$:

Remark: Let us consider for example the wave equation for the initial data u_0 and u_1 and with the simple homogeneous Dirichlet boundary condition:

$$\begin{cases} \partial_t^2 u - \Delta u = F(x, t) & (x, t) \in \Omega \times]0, T[\\ u|_{t=0} = u_0(x), \quad \partial_t u|_{t=0} = u_1(x) \\ u|_{\partial\Omega} = 0, \quad t \in [0, T] \end{cases}$$

Let $(\phi_k, \lambda_k)_{k \in \mathbb{N}}$ be L^2 -orthonormal solutions of the eigenvalue problem of the Dirichlet Laplacian $-\Delta_D$ (auto-adjoint strictly positive operator).

With notations $\omega_k = \sqrt{\lambda_k}$ and $f_k(t) = \int_{\Omega} F(x, t) \phi_k(x) dx$ it holds

$$u(x, t) = \sum_{k=1}^{\infty} \cos(\omega_k t) (u_0, \phi_k)_{L^2(\Omega)} \phi_k + \frac{\sin(\omega_k t) (u_1, \phi_k)_{L^2(\Omega)}}{\omega_k} \phi_k + \int_0^t f_k(\tau) \frac{\sin \omega_k(t - \tau)}{\omega_k} d\tau \phi_k.$$

For $F = u_1 \equiv 0$ we have

$$u(x, t) = \sum_{k=1}^{\infty} \cos(\omega_k t) (u_0, \phi_k)_{L^2(\Omega)} \phi_k.$$

As $(\phi_k)_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$

$$\begin{aligned} \int_{\Omega} |u|^2 dx &= \int_{\Omega} \left(\sum_{k=1}^{\infty} \cos(\omega_k t) (u_0, \phi_k)_{L^2(\Omega)} \phi_k \right)^2 dx \\ &= \sum_{k=1}^{\infty} \cos^2(\omega_k t) (u_0, \phi_k)_{L^2(\Omega)}^2. \end{aligned}$$

We say that $(u_0, \phi_k)_{L^2(\Omega)}^2$ is **the coupling** of the source u_0 with the eigenmode ϕ_k .

To better understand the role of the localized eigenmodes (here on Γ), one can review the associated temporal problem without forgetting that $\text{Re}(\alpha) > 0$ and $\text{Im}(\alpha) < 0$:

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \\ u|_{\Gamma_{Dir}} = 0, \quad \frac{\partial u}{\partial n} \Big|_{\Gamma_{Neu}} = 0, \\ \frac{\partial u}{\partial n} - \frac{1}{\omega} \text{Im}(\alpha(x)) \partial_t u + \text{Re}(\alpha(x)) u|_{\Gamma} = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = 0 \end{cases}$$

which gives (see TD 1), if we work in the space

$$X_0(\Omega) = \left\{ u \in H^1(\Omega) \mid \text{Tr } u|_{\Gamma_{Dir}} = 0 \right\} \times L^2(\Omega)$$

with

$$\|(u, v)\|_{X_0(\Omega)}^2 = \int_{\Omega} (|\nabla_x u|^2 + |v|^2) dx + \int_{\Gamma} \operatorname{Re}(\alpha(x)) |\operatorname{Tr} u|^2 d\mu,$$

that

$$\partial_t \left(\|(u, \partial_t u)\|_{X_0(\Omega)}^2 \right) = -\frac{2}{\omega} \int_{\Gamma} |\operatorname{Im}(\alpha(x))| |\operatorname{Tr} \partial_t u|^2 d\mu.$$

By this formula the decreasing of the energy depends of the values of $\frac{2}{\omega} \int_{\Gamma} |\operatorname{Im}(\alpha(x))| |\operatorname{Tr} \partial_t u|^2 d\mu$.

We notice that this time $-\Delta$ has complex-valued boundary conditions and that it is not an auto-adjoint operator.

Assuming that $(\operatorname{Tr} \phi_k)_{k \in \mathbb{N}}$ is an orthogonal system of $L^2(\Gamma)$ and without detailing the functions which depend only on λ_k and t , we formally obtain that

$$\begin{aligned} \int_{\Gamma} |\operatorname{Tr} \partial_t u|^2 d\mu &= \int_{\Gamma} \left| \sum_{k=1}^{\infty} g(\lambda_k, t) (u_0, \psi_k)_{L^2(\Omega)} \operatorname{Tr} \phi_k \right|^2 d\mu \\ &= \sum_{k=1}^{\infty} |g(\lambda_k, t)|^2 |(u_0, \psi_k)_{L^2(\Omega)}|^2 \int_{\Gamma} |\operatorname{Tr} \phi_k|^2 d\mu. \end{aligned}$$

Therefore, to increase the speed of energy absorption we need to increase

1. the value of the coefficient $(u_0, \psi_k)_{L^2(\Omega)}$ (the coupling with the eigenmode of the adjoint problem with the source, here with the initial data u_0),
2. the value of $\int_{\Gamma} |\operatorname{Tr} \phi_k|^2 d\mu$ (thus favor the localization of the eigenmodes on the boundary Γ).

So, it would be ideal to excite with the source the eigenmodes which are localized on the boundary, or in other words, which have their maximum values on the boundary Γ . The existence and the number of the eigenmodes localized on the boundary Γ depend on its form.

We finalize this chapter by the following empirical conclusion:

1. Geometrical irregularity increases the localization of eigenmodes.
2. Perturbed localized eigenmodes increase energy dissipation.

Chapter 5

Control of the waves

5.1 Control theory: from ordinary to partial differential equations

5.1.1 Controllability of models described by ODEs

In the course “Automatique et contrôle” of D. Dumur (see p. 138 of the coursebook) we can see the following control problem for a process described by a general ordinary differential system or dynamic system

$$\begin{cases} \frac{d}{dt}x(t) = f(x(t), u(t), t), & 0 \leq t \leq T \\ y(t) = \psi(x(t), u(t), t). \end{cases} \quad (5.1)$$

Here

- $y(t) = \begin{pmatrix} y_1(t) \\ \dots \\ y_\ell(t) \end{pmatrix}$ is a function of measurements or observation (actually it also contains the initial data $x(0) = x_0$),
- $u(t) = \begin{pmatrix} u_1(t) \\ \dots \\ u_m(t) \end{pmatrix}$ is a control,
- $x(t) = \begin{pmatrix} x_1(t) \\ \dots \\ x_n(t) \end{pmatrix}$ is the state of the dynamic system.

It is an ordinary differential equation, as all vector functions depend only on one variable, the time t .

The general aim is to find a control $u(t)$ such that the state $x(t)$ satisfies (5.1) (for the given and fixed $y(t)$ and f).

If the state (or simply the solution) x depends not only on time but also from the space variables, then we obtain a process described by a partial differential system. Actually, suppose we discretize the PDE problem on the space variables (by the finite differences or the finite elements). In that case, we obtain the dynamic system of type (5.1) involving only the time variable.

5.1.2 Controllability of models described by PDEs

Let $\Omega \subset \mathbb{R}^n$ be a domain and $x = (x_1, \dots, x_n) \in \Omega$. We consider a state $v(t, x) : [0, T] \times \Omega \rightarrow \mathbb{R}$.

In the control theory for PDEs, there are mainly two types of problems, corresponding to what we require from our controlled solution $v(t, x)$

1. the exact or final controllability,
2. the controllability of moments or the integral overdetermination.

If we take the example of the initial boundary valued problem for the heat equation

$$\begin{cases} \partial_t v - D\Delta v = u(x, t)h(x, t), & (x, t) \in \Omega \times]0, T[, \\ v|_{t=0} = v_0(x), & x \in \Omega, \\ v|_{\partial\Omega} = 0, & t \in [0, T], \end{cases} \quad (5.2)$$

where the coefficient h , the initial temperature distribution v_0 and the diffusion coefficient $D > 0$ are fixed and known, but $u(x, t) : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a **control** which allow to ensure that the solution v satisfies one of the following overdetermination conditions:

1. **the final overdetermination condition**

$$v|_{t=T} = f(x) \quad x \in \Omega \quad (5.3)$$

for a given f .

2. **the integral overdetermination condition:**

$$\int_{\Omega} v(t, x)\omega(x)dx = \chi(t) \quad t \in [0, T], \quad (5.4)$$

presented by the space integral over Ω with $\omega(x)$ and $\chi(t)$ are given functions, or

$$\int_0^T v(t, x)\hat{\omega}(t)dt = \hat{\chi}(x) \quad x \in \Omega, \quad (5.5)$$

presented by the time integral with given functions $\hat{\omega}(t)$ and $\hat{\chi}(x)$.

Actually, if u is known and there are no any condition of overdetermination, then v is the usual solution of the direct problem. But here, we don't want any solution v , but a such solution v , that it satisfies our additional condition (5.3) or (5.4) or (5.5) (as the condition is additional it is named a **condition of overdetermination**). To be able to ensure it, we need to find a control (the function u here). In other words, the control is also an unknown as v .

So, we have

- system (5.2) with (5.3) is the final or exact controllability problem: we are searching a control u which allows at a **final time** T to have **exactly** the heat distribution $f(x)$, knowing that initially for $t = 0$ it was equal to $v_0(x)$. Equivalently, we move our process from the initial state to a final imposed state for a finite time. In many cases, the problem is to move the solution from an initial state to 0 (for a finite time).

- system (5.2) with (5.4) or with (5.5) is a problem of the controllability of moments, where the integral overdetermination could be interpreted as the space or time mean values of the solutions following the evolution in time given by $\chi(t)$ (for instance, for the nuclear reactor it is important to keep the mean temperature constant or very slowly evaluated in time) or to have the same space distribution as $\hat{\chi}(x)$ (for instance, to be mainly localized in one specific subregion of Ω).

Remark: If we discretize problem (5.2) on x , then $v(x, t) \mapsto V(t) = \begin{pmatrix} v_1(t) \\ \dots \\ v_N(t) \end{pmatrix}$ for which the described controllability problems become of the same form as (5.1).
ODEs can be seen as approximations of PDEs.

The other type of controllability problems is the **optimal control problems**, *i.e.* the problems, where we want to find a control u such that the corresponding solution minimizes or maximizes a functional, for example, the energy of the system.

A special case of optimal control problems is the **shape optimization problems**, which we consider in detail here for the Helmholtz equation (see Chapter 7, where χ describing the presence or absence of porous material in the Robin boundary condition is a control). The solution of a PDE, for instance of the Helmholtz equation, v depends on the shape of the domain Ω on which it has been found: for different Ω_1 and Ω_2 the corresponding solutions are different. Thus, the shape of the domain could be a control such that the solution v , found for this shape, minimizes a given functional, for instance, the energy of the wave (see Section 5.4).

5.2 Controllability problems for the propagation of waves

5.2.1 Exact controllability

Let Ω be a rectangular domain in $\mathbb{R}^2$ (see Fig. 5.1). In [39] it was considered the following exact

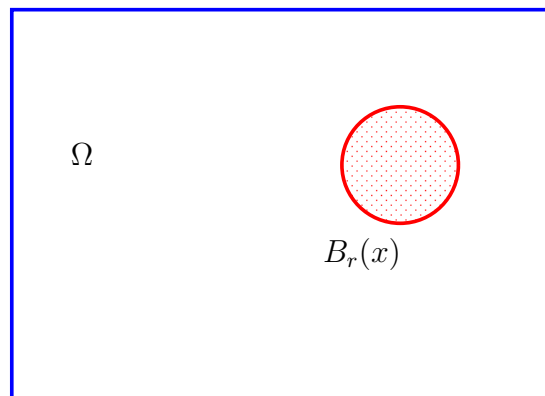


Figure 5.1 – The rectangular domain with a control supported inside on a ball.

controllability problem for the wave equation

$$\begin{aligned} \partial_t^2 u - \Delta u &= h(t, x) \quad]0, T[\times \Omega, \\ u &= 0 \text{ on } [0, T] \times \partial\Omega, \\ u|_{t=0} &= u^0(x) \quad x \in \Omega, \\ \partial_t u|_{t=0} &= u^1(x) \quad x \in \Omega, \\ \boxed{u(T, x) = \partial_t u(T, x) = 0 \quad x \in \Omega}, \end{aligned}$$

where the initial conditions u^0 and u^1 are given and h is the unknown control with a compact support inside Ω . The question is to find h such that at a final time T the wave completely disappears, i.e. $u(T, x) = \partial_t u(T, x) = 0$ for all $x \in \Omega$. Thanks to [39] we have the following theorem

Theorem 5.2.1. *For all $T > 0$ sufficiently large and for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ given, there exists a control $h \in L^2([0, T] \times \Omega)$ with $\text{supp}(h) \subset [0, T] \times \overline{B_r(x)}$.*

5.3 High frequency waves propagation by rays

The most understood mathematically area of the controllability of waves is for the high frequency limit, when the wave propagation is comparable to the propagation of rays with laws of geometrical optic (while for the low or middle frequencies there are no general mathematical results).

Let us cite these several famous (but very difficult and not required in our course) works of Ralston [59], Russell [64], Bardos-Lebeau-Rauch [9] and Bardos-Rauch [10], which are now classical in the control theory of the high frequency wave propagation. We will only mention the main and useful for us ideas.

5.3.1 Problem with a dissipation on a (part of) boundary

We start with following very important and general result for the wave propagation inside a bounded domain with a dissipative part of boundary [64]:

Theorem 5.3.1 (D. L. Russell). *Let Ω be a bounded domain. If there exists a dissipative part of the boundary, then the acoustical energy decrease to 0 for $t \rightarrow +\infty$.*

In the same time we have

Theorem 5.3.2. *The rate of energy decay depends on the geometry of the boundary of Ω .*

The rate of energy decay can be different, as, for instance, it can be arbitrarily small, and it can be exponential. For the most efficient energy absorption, we are interesting to characterize the geometries of $\partial\Omega$ for which it holds the exponential energy decay in time.

In the high frequencies, it is common to consider the wave propagation as the propagation of rays following [59]

Theorem 5.3.3 (J. Ralston). *Wave equations have solutions that are localized near curves $(t, x(t))$ in space-time. The curves are called rays. More precisely,*

$$\forall \epsilon \in]0, 1[\text{ and } T > 0 \text{ there exists a solution so that}$$

the fraction of the energy located at a distance greater than ϵ from the ray is smaller than $1 - \epsilon$ for $0 \leq t \leq T$ (i.e. it is negligibly small).

For $\partial\Omega \in C^\infty$ (or at least C^3 [12]), then [9] it is possible to have **diffractive points** as presented on Fig. 5.2, i.e. the points of the geometry in which the coming ray does not change its direction. These points are to avoid, as in this points the rays do not interact with the boundary in an

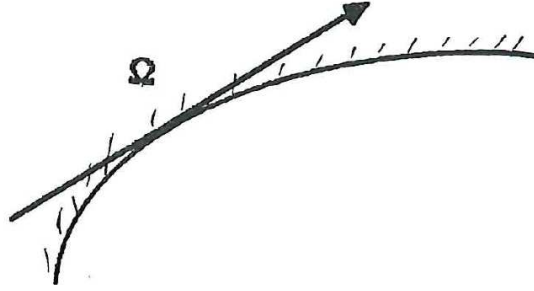


Figure 5.2 – If the ray does not change its direction, then it is a diffractive point [9].

efficient way [9]:

Theorem 5.3.4 (Bardos-Lebeau-Rauch). *To have the exponential decay of the energy it is necessary that any ray meets for $t < T$ a dissipative part of the edge at least once at the non-diffractive point.*

As a corollary of this theorem, for these type of geometry the wave propagation system is exactly controllable to 0 at a (sufficiently large) finite time T , i.e. we have Theorem 5.2.1.

In the same time, for some geometry of the boundary, there are **stable rays** [9]:

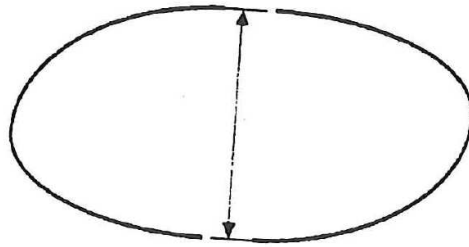


Figure 5.3 – The ray is “captured” by the geometry [9].

Fig. 5.4 give us some examples with a dissipative parts of the boundaries for which it holds the Bardos-Lebeau-Rauch Theorem. The question of non-diffractive points is precised for C^3 boundaries by [12]:

Lemma 5.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth C^3 boundary. Then every ray hits the boundary at a non-diffractive point.*

The Bardos-Lebeau-Rauch Theorem can be generalized also for geometries with “curved” angles [16], but what happens in a fractal boundary case it does not known. Actually it is not clear how to define the direction of the reflected ray by a fractal boundary.

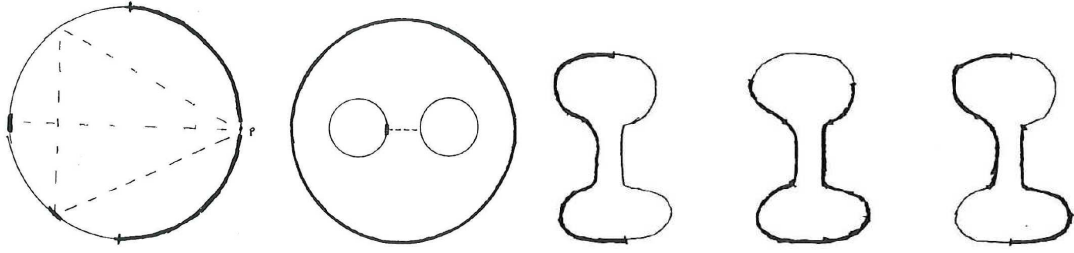


Figure 5.4 – Examples with dissipative parts on the boundary (the thin lines correspond to the non dissipative parts), for which there is an exponential energy decay [9].

5.3.2 Interest of stable periodic rays

If a stable ray occurs between non-dissipative points of the boundaries, then the energy dissipates arbitrarily slowly. But suppose the stable ray is created between dissipative points of the boundary. In that case, this provides the exponential energy decay, since each interaction of the ray with a dissipative surface diminishes the energy. This idea can be expressed as a capture of the wave by a dissipative boundary (see forms of the anechoic chambers on Fig. 2.4). More there are interactions of the ray with the absorbing boundary, more effective absorption is (see, for example, Fig. 5.5).

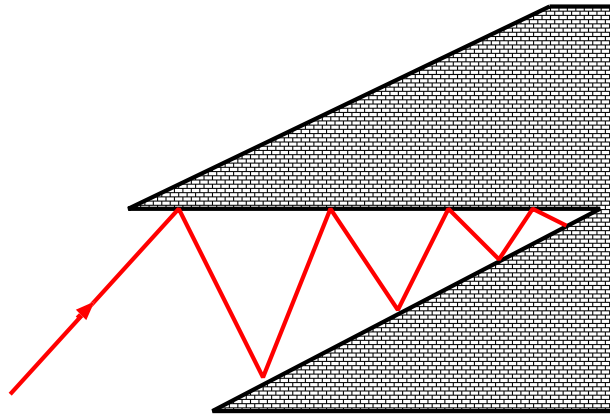


Figure 5.5 – Example of a ray propagating between two absorbing cones, which increases the number of its interaction to compare to the plane absorbing wall.

In [10] it was found the geometrical condition on the boundary shape which creates a captive ray (see Fig. 5.6):

Lemma 5.3.2 (Bardos-Rauch). *Let k_1 and k_2 be the curvatures at the two points of contact, and $2R$ be the distance between the contact points.*

The ray is stable if and only if

$$\left| \frac{(k_1\sigma_2 + k_2\sigma_1)^2 - (k_1^2 + k_2^2)}{2k_1k_2} \right| < 1, \quad \sigma_j = 1 - 2k_jR.$$

For more complicated geometries as on Fig. 5.8 for prefractal examples, there are also results about stable rays [57], which also called billard's orbits.

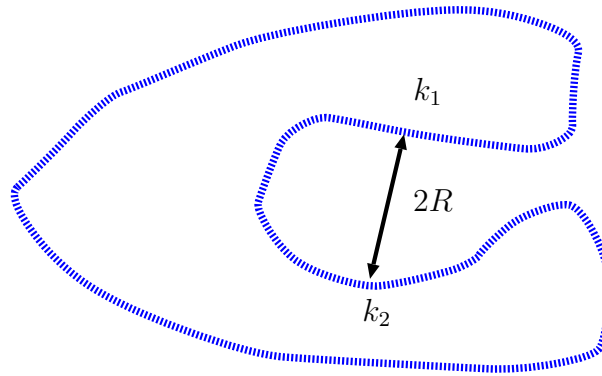


Figure 5.6 – Captive geometry creating the stable periodic rays [10].



Figure 5.7 – The minor axis of an ellipse provide a stable ray and the major axis is unstable. The parallel plane parts of boundary as for the rectangular provide stable rays [10].

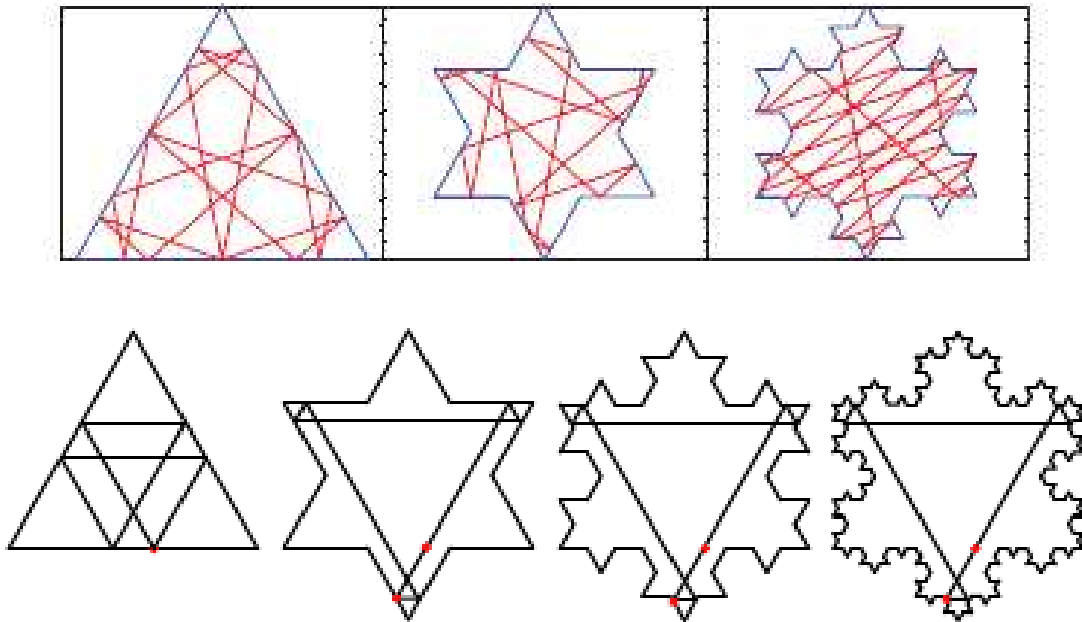


Figure 5.8 – Stable periodic rays in prefractals (billard's orbits) [57].

5.4 Geometrical optimal control for a fixed frequency

Let us suppose that all wall is absorbing, and it is constructed with the same fixed porous material. The question is to find for a fixed frequency of a source an optimal shape of the wall, which minimizes the acoustical energy. The problem is naturally related to different constraints of building such wall, coming from its concrete application, as:

1. what it is the volume of the wall (the question is related with the quantity of the porous material and the final price of the wall),

2. the dimensions of the wall, which could be viewed as the area occupied by the wall (for instance for a barrier for the highways, its shape cannot be too closed to the cars),
3. which is the complicity of the shapes could be constructed by the chosen technology (for example, the demolding method of construction limits much more geometrical possible shapes of the wall than the 3D printing, but in any case, it would be appreciated to have an efficient shape the most simple as possible).

These restrictions on the shape of the wall define the class of admissible shapes of Γ (or equivalently of Ω) on which we are searching to minimize the acoustical energy in the framework of the initial problem Fig. 3.1:

$$U_{ad} := \text{the classe of admissible shapes } \Gamma.$$

Let us fixed the volume V_0 of the wall with a plane shape, which we consider as the initial wall of reference (as a wall with the simplest shape), and the acoustical performances of all new shapes of walls are usually compared to the plane shape of the wall.

Mathematically these practical points could be traduced in the following way (see Fig. 5.9):

1. we need to impose that for all admissible walls its volume kept identical to the initial wall with a plane shape $V_0 = hL$ (see Fig. 5.9; it is rather natural to ask more performances for a fixed quantity of material, making it possible to compare the walls of different shapes with the wall of the plane shape).
2. we need to restrict the area where the shape of the wall can change, denoted by G (*i.e.* for all admissible shapes $\Gamma \subset \overline{G}$).
3. we need to ensure the “simplicity” of the shapes which we consider, *i.e.* to impose that all admissible Γ
 - are of a finite length less or equal to M (uniformly for all Γ , and naturally, the length of Γ is not less than the length of the plane shape Γ_0 .)
 - are (uniformly) Lipschitz and are locally not too complicated: for a fixed constant $\hat{c} > 0$ it holds the following condition

$$\forall x \in \Gamma \quad \int_{\Gamma \cap B_r(x)} d\sigma \leq \hat{c}r. \quad (5.6)$$

5.4.1 Model and admissible shapes

We consider a case of a tunnel with an absorbing barrier (see Fig. 3.7) and model it with the Helmholtz system (5.7) with an absorbing boundary condition

$$\begin{cases} \Delta u + \omega^2 u = f(x) & x \in \Omega, \\ u = g(x) \quad \text{sur } \Gamma_{Dir}, & \frac{\partial u}{\partial n} = 0 \quad \text{sur } \Gamma_{Neu}, \\ \frac{\partial u}{\partial n} + \alpha(x, \omega)u = 0 & \text{sur } \Gamma, \quad \text{Re}(\alpha) > 0 \text{ et } \text{Im}(\alpha) < 0. \end{cases} \quad (5.7)$$

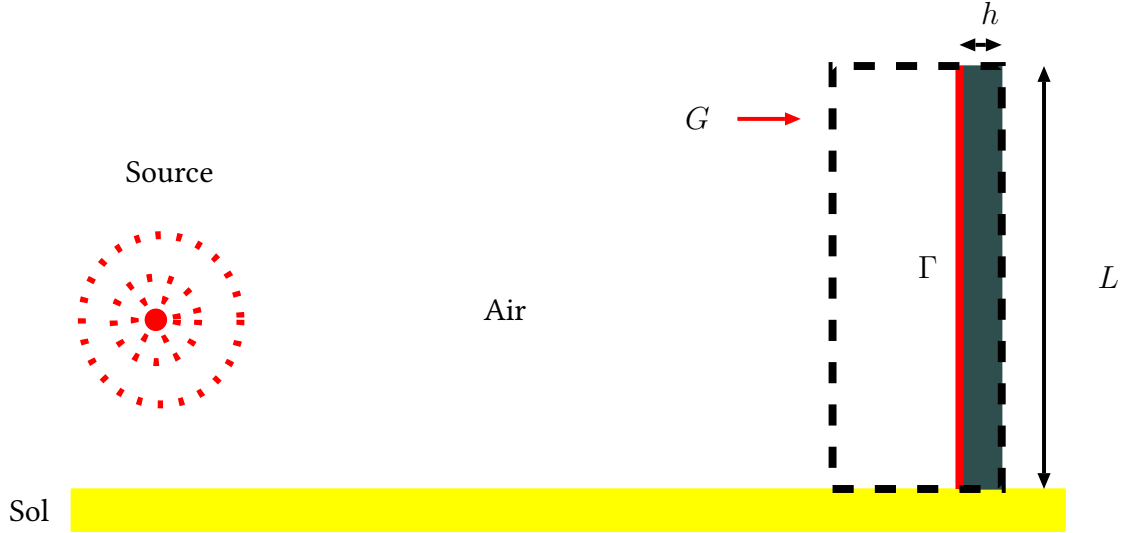


Figure 5.9 – Illustration for the volume and area G constraints on the admissible class of walls.

We consider the two-dimensional shape design problem, which consists of optimizing the shape of Γ with the Robin dissipative condition to minimize the acoustic energy of system (5.7). The boundaries with the Neumann and Dirichlet conditions Γ_{Dir} and Γ_{Neu} are supposed to be fixed.

We also define a fixed open set D with a Lipschitz boundary, containing all domains Ω . Even if we consider the case of Fig. 5.9 we also need to restrict our problem to a global fixed bounded domain D (since even from the numerical point of view is not possible to work in all $\mathbb{R}^n$). Actually, as only a part of the boundary (precisely Γ) changes its shape, we also impose, as mentioned previously, that the changing part always lies inside of the closure of a fixed open set G with a Lipschitz boundary: $\Gamma \subset \overline{G}$. The set G forbids Γ to be too close to Γ_{Dir} , making the idea of an

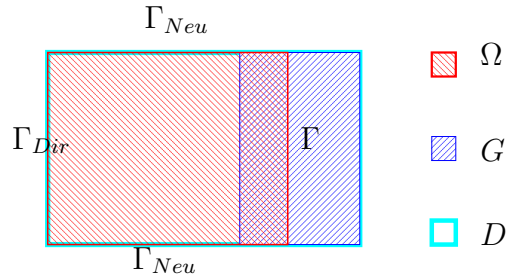


Figure 5.10 – Example of a domain Ω in $\mathbb{R}^2$ with three types of boundaries: Γ_{Dir} and Γ_{Neu} are fixed and Γ can be changed in the restricted area $\overline{G}$. Here $\Omega \cup G = D$ and obviously $\Omega \subset D$.

acoustical wall more realistic (for an example, see Fig. (5.10)).

The admissible class of domains [53] includes all domains, subsets of a fixed Lipschitz domain D and containing Γ in a fixed compact set $\overline{G}$ in D

$$U_{ad}(\Omega_0, \mathcal{Lip}, \hat{c}, M, G) = \{\Omega \in \mathcal{Lip} \mid \Gamma_{Dir} \cup \Gamma_{Neu} \subset \partial\Omega, \Gamma \subset \overline{G}, \\ 0 < \text{Vol}(\Gamma) = \int_{\Gamma} d\sigma \leq M < +\infty, \int_{\Omega} dx = \text{Vol}(\Omega_0)\}, \quad (5.8)$$

where $\mathcal{Lip}$ the class of all domains $\Omega \subset D$ with (uniformly) Lipschitz boundaries of a bounded length $M > 0$ and supposed to be not too complicated by the local condition (5.6).

Let us notice that the values of M , or more precisely its lower bound, influence the minimal acoustical energy value. For instance, naturally, if Γ_0 is the plane shape, then all Γ are such that $\text{Vol}(\Gamma) \geq \text{Vol}(\Gamma_0) > 0$, which gives us a reasonable restriction to impose $M \geq \text{Vol}(\Gamma_0) > 0$: if $M = \text{Vol}(\Gamma_0)$ the unique solution is the plane shape, as soon as it is impossible to relate two fixed points in the plane by a curve shorter than the line. However, the more interesting problem is to take M still finite but “sufficient bigger” than $\text{Vol}(\Gamma_0)$.

5.4.2 Remarks about the existence of an optimal shape

From a numerical point of view, the finite-dimensional optimization problems all times have at least one optimal shape, *i.e.* all times have a solution. It could be understood in the sense that a discretized domain is given by a finite mesh, which defines a finite number of possible changes of its boundary satisfying all desired restriction criteria. Hence, between these finite number of possibilities, it is all times possible to find at least one optimal shape. It is not surprising, as soon as for numerics, the infinite sequences do not exist, and, roughly speaking, all possible nonexistence results and other counter-examples usually come from the passage to the limit.

From a mathematical point of view, when the domains are not discretized and are subsets of $\mathbb{R}^n$ and a considered on it model has a solution from a space of infinite dimension, this question of the existence of an optimal shape is not trivial at all. For instance, we can cite different classical examples of shape optimization problems for which the desired optimal shape does not exist [41, Sec. 4.2, p. 137].

From the theoretical point of view, if we consider the admissible shapes defined in subsection 5.4.1, as (uniformly) Lipschitz boundary with the usual Lebesgue measure, then for the shape optimization problem

$$\inf_{\Omega \in U_{ad}(\Omega_0, \mathcal{L}ip, \hat{c}, M, G)} J(\Omega, u(\Omega))$$

there exists an optimal shape $\Omega_{opt} \in U_{ad}(\Omega_0, \mathcal{L}ip, \hat{c}, M, G)$ providing the infimum of J (it also could provide a minimum, but as soon as we don't know sufficiently the properties of the minimizing sequence of $(\Omega_k)_{k \in \mathbb{N}^*} \subset U_{ad}(\Omega_0, \mathcal{L}ip, \hat{c}, M, G)$ is not possible to conclude it). This infimum is realized for the weak solution of the Helmholtz problem on $\Omega_{opt} \in U_{ad}(\Omega_0, \mathcal{L}ip, \hat{c}, M, G)$, *i.e.* on a domain with a Lipschitz shape Γ_{opt} , but with possibly different measure on it (a measure equivalent to the Lebesgue measure).

The problem occurs from the fact that the uniformly bounded (by M) perimeters are not continuous (only lower continuous [41]): if $\Gamma_k \rightarrow \Gamma_{opt}$ (actually in the Hausdorff sense), this does not imply that

$$\forall \phi \in C(\overline{D}) \quad \int_{\Gamma_k} \phi d\sigma \rightarrow \int_{\Gamma_{opt}} \phi d\sigma,$$

and the only possibility that we have is to say that there exists a finite measure μ^* on Γ_{opt} , such that

$$\forall \phi \in C(\overline{D}) \quad \int_{\Gamma_k} \phi d\sigma \rightarrow \int_{\Gamma_{opt}} \phi d\mu^*.$$

Suppose $\mu^*(\Gamma_{opt}) = \int_{\Gamma_{opt}} 1 d\sigma$ (unfortunately not guaranteed). In that case, we can establish the existence of the minimum of our optimization problem, since we stay in the same admissible class with the same fixed usual Lebesgue measure. Otherwise, and it is the general situation, we can only conclude the existence of an optimal shape Ω_{opt} realizing the infimum of J on $U_{ad}(\Omega_0, \mathcal{L}ip, \hat{c}, M, G)$. For more details (not required in the framework of this course), see [53].

It is similar to the situation of the parametric optimization, where it is not generally possible to find its minimum on $U_{ad}(\beta)$, but only its infimum. A relaxation problem consists of allowing different measures on the boundary Γ and passing from the uniformly bounded perimeter shapes to the shapes with uniformly bounded measures. Roughly speaking, this can be viewed as a class of **shapes with bounded dimensions** $n - 1 \leq d < n$, involving d -dimensional fractals, Lipschitz boundaries, and their mixtures and other generalizations [43, 44]. In this class of admissible shapes, there is the existence of the minimum of J .

5.5 Non unicity of an optimal shape

For a fixed frequency and even on a fixed range of frequencies, different absorbing shapes can provide the same values of energy.

It comes [31, 33, 34] from the identical spectral properties for different shapes, *i.e.* the eigenvalues.

For example, Fig. 5.11 shows two domains for which $-\Delta$ has the same eigenvalues. We don't

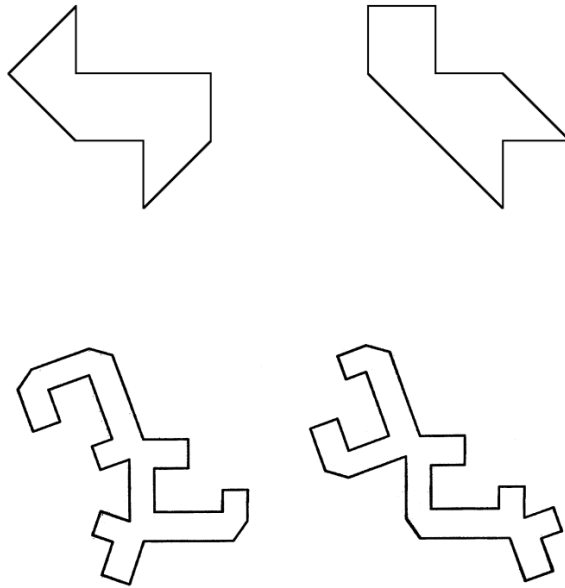


Figure 5.11 – Carolyn Gordon, David Webb and Scott Wolpert (1992): two examples of domains with the same spectrum.

have the uniqueness of the optimal shape, since different shapes can have the same spectrum and be identically efficient in the dissipation of the energy in the fixed range of frequency.

We can illustrate it numerically on Fig. 5.12. We take as a reference domain $\Omega^{\text{flat}} =]0, 2[\times]0, 1[$ with a flat boundary Γ_0 fixed at $x = 2$. The characteristic lengths of Ω^{flat} are $\ell = 1$ and $L = 2\ell$. The Helmholtz equation is considered with a wave number $k = \frac{\omega}{c_0}$, *i.e.*,

$$\Delta u + k^2 u = -f,$$

where c_0 is the sound speed in air. We take

$$f = 0, \quad g = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y - 1/2)^2}{2\sigma^2}\right)$$

with $\sigma = 1$ in the Helmholtz boundary value problem. For the chosen σ , the smallest wavelength, excited by g , is $\lambda = \frac{\ell}{2}$. The parameter α in the Robin boundary condition depends on the value of the frequency ω . The parameter α is calculated for ISOREL porous medium, using a minimization of the difference between the solution of the problem with a volume dissipation (described by a damped wave equation) and the solution of the problem with the boundary dissipation for the flat shape boundary (see Theorem B.1 [53]). We solve the Helmholtz boundary value problem on a fine mesh with the size $h = \frac{\ell}{64}$ (see Appendix B). For waves with the wavelength equal to $\frac{\ell}{2}$, this typically gives dispersion errors of the order 10^{-3} , since the dispersion error due to the centered finite difference approximation of the Laplacian is known to be $\frac{(kh)^2}{24}$, with $kh = \frac{2\pi}{32}$ here. We fix the frequency $\omega_0 = 3170$, which is a local maximum of

$$J(\omega, \Omega) = \int_{\Omega} |u|^2 dx,$$

calculated for $\Omega^{\text{flat}} =]0, 2[\times]0, 1[$ in a range of frequencies $\omega \in [3000, 6000]$.

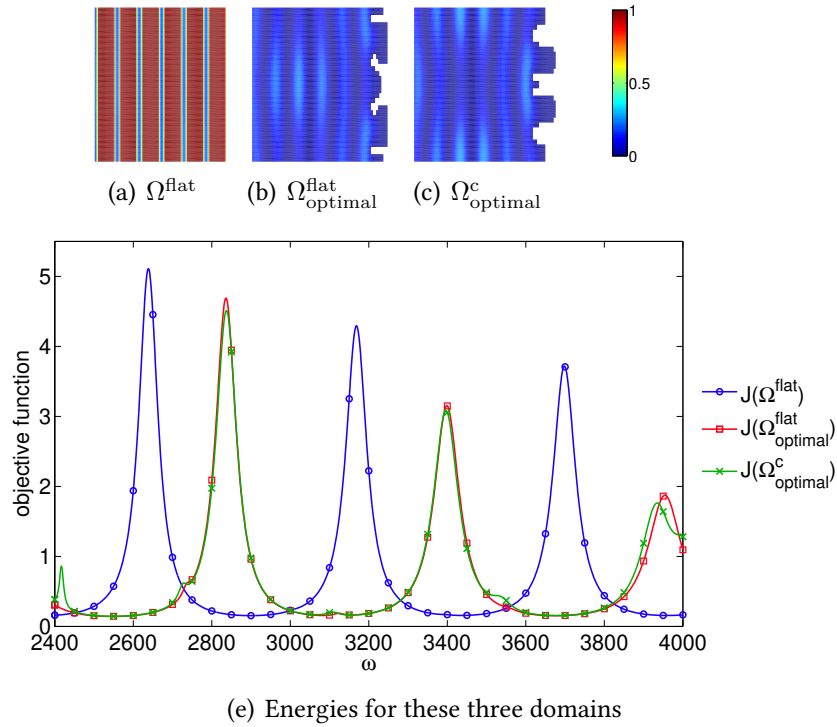


Figure 5.12 – The values of $|u|^2$ are presented on the domain with the flat boundary shape and two optimal domains for the fixed frequency $\omega_0 = 3170$. Numerical values of $J(\omega, \Omega) = \|u(\omega, \Omega)\|_{L^2}^2$ at $\omega_0 = 3170$ are almost the same: $J(\Omega_{\text{opt}}^{\text{flat}}) = 0.1654$ and $J(\Omega_{\text{opt}}^c) = 0.1659$. The graphs of $J(\omega, \Omega_{\text{optimal}}^{\text{flat}})$ and $J(\omega, \Omega_{\text{optimal}}^c)$ are almost the same for the chosen range of frequencies.

5.6 Engineering acoustical energy control on a range of frequencies

From the engineering point of view the most important question is to find “range-optimal” shape, *i.e.* a shape of an absorbing wall, the most efficient on a fixed range of frequencies. An

other optimality could be related with the geometrical complexity and ability to be constructed.

Numerically we start by finding of a “range-optimal” shape in absorption and after we simplify it and compare two performances.

Once we know how to find numerically an optimal shape for one fixed frequency, we can develop an algorithm of the energy minimization on a range of frequencies. For the Helmholtz model

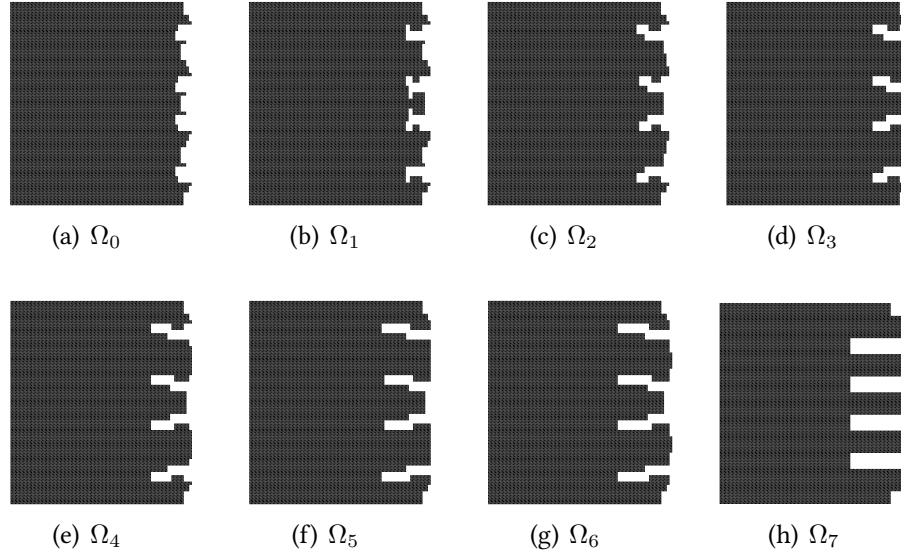


Figure 5.13 – Shapes, which are used in the optimization algorithm process: from left to right in the top line - Ω_0 (the initial shape), Ω_k , $k = 1, 2, 3$, and from left to right in the bottom line - Ω_k , $k = 4, 5, 6, 7$. The domain Ω_7 is generated manually in the aim to simplify Ω_6 (the final “range-optimal” shape).

described above, we minimize the acoustical energy $J(\omega, \Omega) = \int_{\Omega} |u(\omega, \Omega)|^2 dx$ in the range $\omega \in [3000, 6000]$.

As previously, we fix the frequency $\omega_0 = 3170$ of a local maximum of J on $\Omega_{\text{flat}} =]0, 2[\times]0, 1[$. We perform the shape optimization algorithm for this frequency, taking as the initial shape Ω_0 , given on Fig. 5.13, and we obtain Ω_1 , optimal at $\omega = 3170$. Noticing that all local maxima of $J(\Omega_1)$ are smaller than the local maxima of $J(\Omega_{\text{flat}})$ (see Fig. 5.14), we choose Ω_1 as the initial domain and restart the optimization algorithm, minimizing in the neighborhood of Ω_1 the sum of functionals $\sum_{k=1}^3 J(\Omega)(\omega_k)$, where $\omega_1 = 3410$, $\omega_2 = 4025$ and $\omega_3 = 4555$ are the local maxima of $J(\Omega_1)$. This minimization gives the optimal shape Ω_2 , such that

1. Ω_2 is “range-optimal” in the neighborhood of ω_k for $k = 0, 1, 2, 3$;
2. all local maxima of $J(\Omega_2)$ are smaller than the local maxima of $J(\Omega_1)$.

Choosing $\omega_4 = 3625$ and $\omega_5 = 4240$, corresponding to the local maxima of $J(\Omega_2)$, we take Ω_2 as the initial domain and restart the optimization algorithm, minimizing $J(\Omega)(\omega_4) + J(\Omega)(\omega_5)$ to obtain the optimal shape Ω_3 , such that

1. Ω_3 is “range-optimal” in the neighborhood of ω_k for $k = 0, \dots, 5$;
2. all local maxima of $J(\Omega_3)$ are smaller than the local maxima of $J(\Omega_2)$.

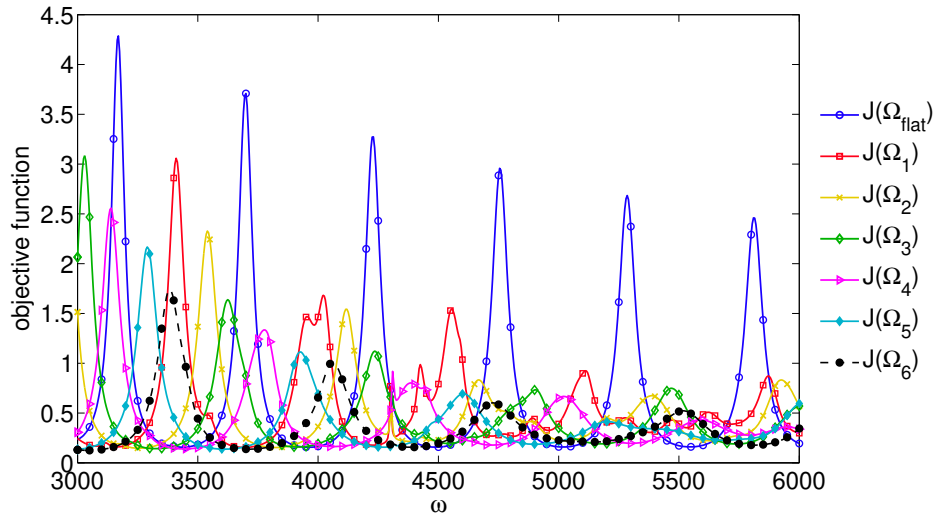


Figure 5.14 – The values of the objective function $J(\Omega_0)$ ($A = 1, B = 0, C = 0$) for the flat shape as a function of $\omega \in [3000, 6000]$ are presented by the line with circles, the values of $J(\Omega_1)$ (see Fig. 5.13 for the shape of Ω_1) are presented by the line with squares, of the values $J(\Omega_2)$ by the line with stars, those of $J(\Omega_3)$ by the line with empty rhombus, those of $J(\Omega_4)$ by the line with arrows, those of $J(\Omega_5)$ by the line with full rhombus, and those of $J(\Omega_6)$ by the black dashed line.

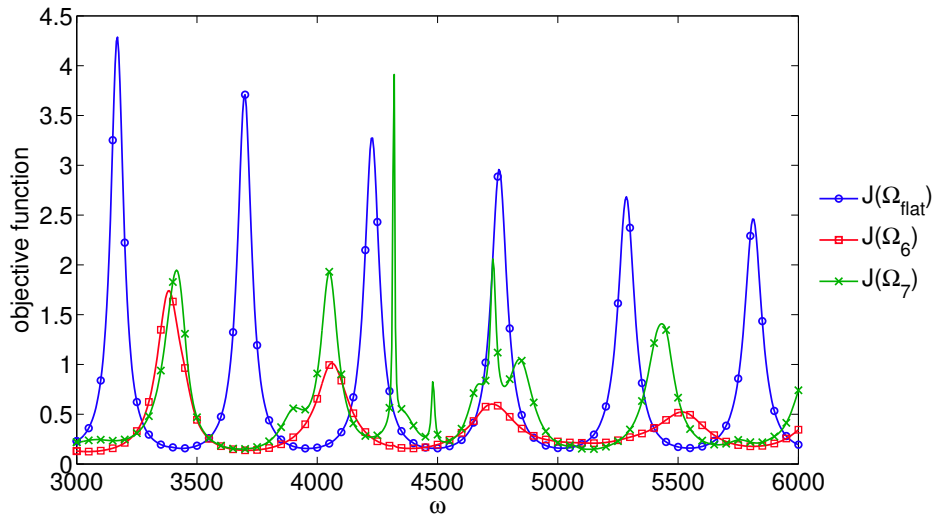


Figure 5.15 – Comparison of the dissipative properties of the flat shape Ω_{flat} , the optimal Ω_6 and of its simplification Ω_7 . The values of $J(\Omega_{\text{flat}})$, of $J(\Omega_6)$ and of $J(\Omega_7)$ ($A = 1, B = 0, C = 0$) as functions of $\omega \in [3000, 6000]$ are given by the lines with circles, squares and stars respectively.

We iterate this process up to Ω_6 , and we are stopped by the restriction that Γ must be contained by the area $\overline{G} = [\frac{3}{2}, 3] \times [0, 1]$.

The shape of Ω_6 contains multiscale details, which ensures the dissipative performances of the wall in a large range of frequencies (see Fig. 5.14). Thinking about the demolding process of wall construction, we simplify the geometry of Ω_6 , deleting the multi scales, and keeping only the largest characteristic scale of Ω_6 (see the domain Ω_7 , generated by hand, on Fig. 5.13). As we

can see from Fig. 5.15, since we have kept almost unchanged the largest characteristic geometric size for Ω_6 and Ω_7 , the energy dissipation is also almost the same in the corresponding range of frequencies (see red and green lines for $[3000, 3700]$ on Fig. 5.15). As all smaller-scale details have been deleted, the shape of Ω_7 is not as good as the shape of Ω_6 to dissipate higher frequencies (see lines with squares and stars for $[3700, 6000]$ on Fig. 5.15). Hence, Fig. 5.15 shows that the compromises between two desired properties “to be the most dissipative” (as Ω_6 here) and “to be simple to construct” (on the example of Ω_7) is not too bad, especially if we know the most critical frequencies to dissipate.

Chapter 6

Compactness

6.1 Compact sets in the normed vector spaces

6.1.1 Definitions and examples

Definition 6.1.1. Let $(X, \|\cdot\|_X)$ be a normed vector space. A set $M \subset X$ is called compact, if any sequence of M contains a sub sequence converging to an element of M :

$$\forall (x_n)_{n \in \mathbb{N}} \subset M \quad \exists (x_{n_k})_{k \in \mathbb{N}} \subset (x_n)_{n \in \mathbb{N}} \quad \text{and} \quad \exists x \in M \text{ such that} \\ x_{n_k} \rightarrow x \text{ in } X, \quad \text{i.e. } \|x_{n_k} - x\|_X \rightarrow 0, \quad k \rightarrow +\infty.$$

In particular for the finite dimensional spaces there is the Theorem of Heine-Borel

Theorem 6.1.1. (Heine-Borel) In $\mathbb{R}^n$ (or $\mathbb{C}^n$) a set is compact if and only if it is bounded and closed.

Attention: If the space $(X, \|\cdot\|_X)$ has an infinite dimension ($\dim X = +\infty$), as for example the spaces $L^2(\Omega)$, $H^1(\Omega)$ and $C(\overline{\Omega})$, the condition

$$\text{“bounded + closed”} \Rightarrow \text{compact}$$

does not true anymore. We only have the statement

$$\text{compact} \Rightarrow \text{“bounded + closed”}.$$

The following example is actually a very important theorem:

Example 6.1.1. Let $(X, \|\cdot\|_X)$ be a normed linear vector space of infinite dimension. Then the closed unit ball $\overline{B_1}(0) = \{x \in X \mid \|x\|_X \leq 1\}$ is not compact.

But it is more easy to prove it in the case of a Hilbert space:

Example 6.1.2. Let H be a separable Hilbert space (there exists a countable orthonormal basis) of infinite dimension. Let us denote this basis $(e_n)_{n \in \mathbb{N}^*} \subset H$. By its definition

$$(e_k, e_m)_H = \delta_{k,m} = \begin{cases} 1, & k = m \\ 0, & k \neq m. \end{cases}$$

Thus we compute

$$\|e_k - e_m\|_H^2 = (e_k - e_m, e_k - e_m)_H = 2,$$

which means that the distance

$$d_H(e_k, e_m) = \|e_k - e_m\|_H = \sqrt{2} \quad \forall k \neq m.$$

In other words, the elements e_k and e_m are all separated by the distance equal to $\sqrt{2}$. Thus this sequence cannot have any convergent sub sequence in H (there are no an accumulation point). Thus we have taken a sequence of the closed unit ball of H and it dose not have a convergent sub sequence. Thus the closed unit ball in H is not compact.

Let us now give an example of compact sets:

Example 6.1.3. Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence in a normed vector space $(X, \|\cdot\|_X)$ (i.e. there exists $x \in X$ such that $x_n \rightarrow x$ for $n \rightarrow +\infty$ in X). This sequence $(x_n)_{n \in \mathbb{N}}$ with x forms a compact set.

More generally, the compact sets cannot contain any non trivial closed ball of X (these kind of sets are called **nowhere dense** sets in X).

Definition 6.1.2. (Strong convergence) Let $(X, \|\cdot\|_X)$ be a normed linear vector space. We say that a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ converges **strongly** to $x \in X$ if it converges by the norm of X ($x_n \rightarrow x$ in X if $\|x_n - x\|_X \rightarrow 0$ for $n \rightarrow +\infty$).

So, we have seen that the closed ball in the infinite dimensional vector normed space is not compact for the strong type of convergence.

Question: Is it possible to make $\overline{B_r}(x)$ compact in (an infinite dimensional) Hilbert space?

To do it, we need to consider the weak convergence.

6.1.2 Weak convergence in a Hilbert space

In this subsection we suppose to work in a Hilbert space of infinite dimension.

Definition 6.1.3. (Weak convergence) Let H be a Hilbert space. We say that $(x_n)_{n \in \mathbb{N}} \subset H$ converges **weakly** to $x \in H$ and write $x_n \rightharpoonup x$ in H if

$$\forall v \in H \quad (x_n, v)_H \rightarrow (x, v)_H, \quad (6.1)$$

where, as previously, $(\cdot, \cdot)_H$ is an inner product in H .

Attention: The sequence $((x_n, v)_H)_{n \in \mathbb{N}} \subset \mathbb{C}$ is a numerical sequence.

Properties of the weak convergence

Let us now consider the properties of the weak convergence:

Proposition 6.1.1. *Weak limit is unique.*

Proof

Let us suppose the converse. Let (x_n) be a sequence in H such that

$$x_n \rightharpoonup x \quad \text{and} \quad x_n \rightharpoonup \tilde{x} \quad n \rightarrow \infty.$$

Then for all $v \in H$,

$$(x_n, v)_H \rightarrow (x, v)_H, \quad \text{and} \quad (x_n, v)_H \rightarrow (\tilde{x}, v)_H.$$

Thus, the numerical sequence $((x_n, v)_H)_{n \in \mathbb{N}}$ is convergent in $\mathbb{C}$ (or $\mathbb{R}$). Consequently, there exists only one limit of this numerical sequence:

$$\forall v \in H \quad (x, v)_H = (\tilde{x}, v)_H.$$

Therefore,

$$\forall v \in H \quad (v, x - \tilde{x})_H = 0.$$

It means that the vector $x - \tilde{x}$ is orthogonal to all space H , including itself. From $x - \tilde{x} \perp x - \tilde{x}$ it follows that $x - \tilde{x} = 0$. The contradiction. $\square$

Proposition 6.1.2. *The strong convergence implies the weak convergence, but in the infinite dimensional case generally the weak convergence does not implies the strong one:*

$$\boxed{\longrightarrow \Rightarrow \rightharpoonup}, \quad \text{but} \quad \boxed{\rightharpoonup \not\Rightarrow \longrightarrow}.$$

However, if $\dim H = m < \infty$, then (H is isomorph to $\mathbb{R}^m$ or $\mathbb{C}^m$) the weak convergence implies the strong convergence and thus two convergences are equivalent.

Proof

Let us prove $\boxed{\longrightarrow \Rightarrow \rightharpoonup}$. If $x_n \rightarrow x$ in H , then $\|x_n - x\|_H \rightarrow 0$ in H and hence, by the Cauchy-Swartz inequality, for all $v \in H$

$$|(x_n, v)_H - (x, v)_H| = |(x_n - x, v)_H| \leq \|v\|_H \|x_n - x\|_H \rightarrow 0.$$

Then, by Definition 6.1.3, we conclude that $x_n \rightharpoonup x$ in H .

Let us now give an example of a sequence which converges weakly but not strongly in an infinite dimensional Hilbert space.

Let (e_k) be orthonormal basis in H . Then every vector can be approximated by its Fourier series:

$$\forall x \in H \quad x = \sum_{k \in \mathbb{N}} (x, e_k)_H e_k, \quad \text{i.e.} \quad \|x - \sum_{i=1}^k (x, e_k)_H e_k\|_H \rightarrow 0 \text{ for } k \rightarrow +\infty.$$

Here $((x, e_k)_H)_{k \in \mathbb{N}}$ are the coefficients of Fourier of x corresponding to the basis $(e_k)_{k \in \mathbb{N}}$. We recall that it holds the Parseval identity:

$$\forall x \in H \quad \|x\|_H^2 = \sum_{k \in \mathbb{N}} |(x, e_k)_H|^2,$$

which actually means that the numerical series $\sum_{k \in \mathbb{N}} |(x, e_k)_H|^2$ converges. Then

$$\forall x \in H \quad \langle x, e_k \rangle \rightarrow 0 \quad k \rightarrow \infty.$$

Consequently, $e_k \rightharpoonup 0$. But if $n \neq m$ $\|e_n - e_m\|^2 = \langle e_n - e_m, e_n - e_m \rangle = 2$, thus (e_k) is not a Cauchy sequence in H and then $(e_k)_{k \in \mathbb{N}}$ does not converge.

The equivalence of the weak and strong convergences in a finite dimensional space follows from the fact that all norms in this case are equivalent. We omit the proof. $\square$

Proposition 6.1.3. *If $x_n \rightharpoonup x$ in H , then $(x_n)_{n \in \mathbb{N}}$ is bounded (by norm) in H :*

$$\exists C > 0 : \quad \|x_n\|_H < C \quad \text{and} \quad \|x\|_H \leq \liminf \|x_n\|_H,$$

where $\liminf \|x_n\|_H = \lim_{n \rightarrow \infty} (\inf_{m \geq n} \|x_m\|_H)$.

Instead of the proof let us give an example:

Example 6.1.4. *Let us consider $L^2([0, 1])$ and*

$$x_n(t) = \sqrt{2} \sin(\pi n t).$$

The sequence (x_n) is an orthonormal basis of $L^2([0, 1])$. We have

$$\|x_n\| = 1 \quad \Rightarrow \quad \lim_n \|x_n\| = 1.$$

For $\alpha(t) \in L^2([0, 1])$ we define, with notation c_n for Fourier coefficients,

$$f(x_n) = \sqrt{2} \int_0^1 \alpha(t) \sin(\pi n t) dt = \sqrt{2} c_n$$

such that

$$f(x_n) \rightarrow 0, \quad n \rightarrow \infty, \quad \text{and} \quad x_n \rightharpoonup 0.$$

Thus, the limit $x = 0$ and

$$\|x\| = 0 < 1 = \lim_n \|x_n\|.$$

In addition we have the following characterization of the weak convergence:

Proposition 6.1.4. *A sequence $x_n \rightharpoonup x$ in H for $n \rightarrow +\infty$ if and only if*

1. $(\|x_n\|)_{n \in \mathbb{N}}$ is bounded,
2. For $n \rightarrow +\infty$, $(v, x_n) \rightarrow (v, x) \quad \forall v \in E, \quad \overline{E} = H$.

We know that the inner product is continuous by the strong convergence in H :

$$x_n \rightarrow x, \quad y_n \rightarrow y \quad \Rightarrow \quad (x_n, y_n) \rightarrow (x, y).$$

If $x_n \rightharpoonup x$, $y_n \rightharpoonup y$, it does not imply that $(x_n, y_n) \rightarrow (x, y)$. Let us give an example:

Example 6.1.5. Let (e_n) be an orthonormal sequence in H and $x_n = y_n = e_n$. Then $e_n \rightharpoonup 0$ and

$$(e_n, e_n) = \|e_n\|^2 = 1 \not\rightarrow 0 = (0, 0).$$

Proposition 6.1.5. Let H be a Hilbert space. If $x_n \rightarrow x$ and $y_n \rightharpoonup y$ in H , then

$$(x_n, y_n) \rightarrow (x, y).$$

Proof. Since $y_n \rightharpoonup y$ in H , for all $n \in \mathbb{N}$ the norms $\|y_n\|$ are bounded. Let

$$M = \sup_n \|y_n\|.$$

Then, by the Cauchy-Schwartz inequality, we have

$$|(x_n, y_n) - (x, y)| \leq |(x_n - x, y_n)| + |(x, y_n - y)| \leq M\|x_n - x\| + |(x, y_n - y)|.$$

Since $y_n \rightharpoonup y$ in H , then $|(x, y_n - y)| \rightarrow 0$, and since $x_n \rightarrow x$, then $\|x_n - x\| \rightarrow 0$. Therefore, we conclude that $|(x_n, y_n) - (x, y)| \rightarrow 0$. $\square$

Remark: Hence, it is important to have at least one of two sequence converging strongly. If we have two weak convergent sequences, it is not enough to have the convergence of the corresponding inner products.

The following proposition is really very useful.

Proposition 6.1.6. Let H be a real Hilbert space. Let $(x_n)_{n \in \mathbb{N}}$ be a weakly converging sequence to x . Further assume that $\|x_n\|$ converges to $\|x\|$ for $n \rightarrow +\infty$. Then $(x_n)_{n \in \mathbb{N}}$ converges strongly to x .

Proof. We have

$$(x_n - x, x_n - x) = (x_n, x_n) - (x_n, x) - (x, x_n) + (x, x).$$

For $n \rightarrow \infty$, we find that

$$\lim_{n \rightarrow \infty} (x_n - x, x_n - x) = 0,$$

which means that $\|x_n - x\| \rightarrow 0$, i.e. $x_n \rightarrow x$ in H . $\square$

Weak compactness of the closed balls

Theorem 6.1.2. Let H be a Hilbert space. Then the closed non-trivial ball $\overline{B}_r(0)$ is weakly compact:

$$\forall (x_n)_{n \in \mathbb{N}} \subset \overline{B}_r(0) \quad \exists (x_{n_k})_{k \in \mathbb{N}} \subset (x_n)_{n \in \mathbb{N}} : \quad \exists x \in \overline{B}_r(0) \text{ such that } x_{n_k} \rightharpoonup x \quad k \rightarrow +\infty.$$

Attention: As a direct corollary of this theorem we have that all bounded sequence contains a weakly convergent subsequence:

$$\|x_n\| \leq C \quad \forall n \quad \Rightarrow \quad \exists (x_{n_k})_{k \in \mathbb{N}} \subset (x_n)_{n \in \mathbb{N}} : \quad x_{n_k} \rightharpoonup x \text{ in } H.$$

6.2 Compact operators

Definition 6.2.1. A set $M \subset Y$ of a normed vector space $(Y, \|\cdot\|_Y)$ is called **relatively compact** set (in French **pré-compact**), if its closure $\overline{M}$ in Y is a compact set in $(Y, \|\cdot\|_Y)$.

Definition 6.2.2. Let X and Y be Banach spaces. A linear operator $A : X \rightarrow Y$ is **compact**, if it maps all bounded sets of the space X to a relatively compact set of the space Y .

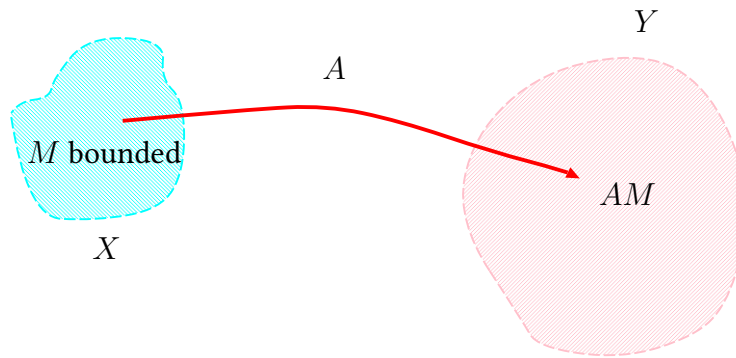


Figure 6.1 – Operator A maps a bounded set M of X in a set AM in Y . If AM is bounded, then A is bounded/continuous operator. If AM is relatively compact then A is compact operator (and also bounded).

Let us denote $B_1 = \overline{B_1(0)} = \{f \in X \mid \|f\|_X \leq 1\}$ the closed unit ball in X .

Remark: The operator A is compact iff AB_1 is a relatively compact set in Y .

Indeed, if A is compact operator, then obviously AB_1 is a relatively compact set in Y .

Conversely, let AB_1 be a relatively compact set in Y . For all bounded sets M in X there exist $r > 0$, the radius of the ball including M :

$$M \subset B_r = rB_1.$$

Consequently, using the linearity of A ,

$$AM \subset rAB_1.$$

Since AB_1 is a relatively compact set, thus rAB_1 too, what implies that AM is relatively compact (any subset of a relatively compact set is relatively compact itself).

Remark: Let us denote by $K(X, Y)$ the set of all compact operators from X to Y (X and Y are Banach spaces). We also recall that $\mathcal{L}(X, Y)$ is the normed vector space of all bounded operators from X to Y with a norm

$$\begin{aligned}\|A\|_{\mathcal{L}(X, Y)} &= \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X \leq 1} \|Ax\|_Y = \sup_{\|x\|_X = 1} \|Ax\|_Y \\ &= \inf\{C \in \mathbb{R}^+ \mid \|Ax\|_Y \leq C\|x\|_X \ \forall x \in X\}.\end{aligned}$$

We notice that

- All compact operators from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$ are bounded operators:

$$K(X, Y) \subset \mathcal{L}(X, Y).$$

Indeed, it is sufficient to notice that if AB_1 is relatively compact, then AB_1 is bounded.

- If $\dim X < \infty$ or $\dim Y < \infty$, then the compactness of the operators is equivalent to the boundness:

$$K(X, Y) = \mathcal{L}(X, Y).$$

Indeed, let $\dim X < \infty$, then AX is finite dimensional. In addition, $AB_1 \subset AX$ and all bounded set in AX is relatively compact in Y . If $\dim Y < \infty$, we have $AB_1 \subset Y$ and there is no difference between the relatively compactness and the boundness in the final dimensional space.

Example 6.2.1. The dual space $X^* = \mathcal{L}(X, \mathbb{R}) = K(X, \mathbb{R})$, i.e. all linear continuous functionals on X are compact operators from X to $\mathbb{R}$.

Attention: M is a relatively compact set in a normed vector space of finite dimension if and only if M is a bounded set.

Theorem 6.2.1. Let X and Y be Banach spaces. The set of all compact operators from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$, denoted by $K(X, Y)$, is a closed vector subspace of $\mathcal{L}(X, Y)$:

1. A is a compact operator $\Rightarrow \forall \lambda \in \mathbb{C} \quad \lambda A$ is compact operator.
2. A and S are compact operators $\Rightarrow A + S$ is compact operator.
3. (A_n) is a sequence of compact operators such that $A_n \rightarrow A$ in $\mathcal{L}(X, Y)$ $\Rightarrow A$ is compact operator.

In addition, $K(X, X)$ forms a closed ideal in $\mathcal{L}(X, X)$, i.e if A is a compact operator and S is a bounded operator, then the operators AS and SA are compact.

Proof

The first statement is admitted without proof. Let now prove

$A \in K(X, X), S \in \mathcal{L}(X, X) \Rightarrow AS$ and SA are compact operators.

By AS and SA we understand $A \circ S$ and $S \circ A$ respectively. Since S maps bounded sets to bounded sets, we have that there exists $r > 0$ such that

$$(AS)B_1 = A(SB_1) \subset AB_r,$$

and since A is compact and B_r is bounded, thus AB_r is relatively compact. In other hand, $(SA)B_1 = S(AB_1)$, where AB_1 is relatively compact. Since $S \in \mathcal{L}(X, X)$, S is continuous and thus preserve the property of the relatively compactness. $\square$

Corollary 6.2.1. *A compact operator A mapping a Banach space X into itself cannot have a bounded inverse A^{-1} if X is infinite-dimensional.*

Proof

If A^{-1} were bounded, then, by Theorem 6.2.1, the identity operator $I = A^{-1}A$ would be compact. But the unit ball is not relatively compact in an infinite-dimensional Banach space. $\square$

We also mention the following property of the compact operators:

Proposition 6.2.1. *Let X be a Banach space and $A \in \mathcal{L}(X, X)$ a compact operator. Then $\text{Im}(I - A)$ is a closed subspaces in X . Here, by I is denoted the identity operator.*

Proof

Let (y_n) belong to $\text{Im}(I - A)$. Then for all $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$x_n - Ax_n = y_n.$$

Let $y_n \rightarrow y$ in X for $n \rightarrow \infty$. Let us prove that $y \in \text{Im}(I - A)$. We consider the following cases:

1. Let (x_n) be bounded in X . Then the set (Ax_n) is relatively compact. Moreover, since

$$x_n = y_n + Ax_n,$$

where (y_n) converges and (Ax_n) is relatively compact, we conclude that (x_n) is also relatively compact (by the composition of a continuous function $x + y$ with a bounded and a compact operators). Thus, there exists a subsequence (x_{n_k}) which converges in X . We denote its limit by $x \in X$. We pass to the limit for $n_k \rightarrow \infty$ in the equation

$$x_{n_k} - Ax_{n_k} = y_{n_k}$$

and obtain, using the continuity of A , that

$$x - Ax = y,$$

i.e. $y \in \text{Im}(I - A)$.

2. Let (x_n) be unbounded in X .

Let V be a subspace of X equal to $\text{Ker}(I - A)$:

$$V = \text{Ker}(I - A) = \{z \in X \mid z - Az = 0\}.$$

We define the distance

$$d_n = d(x_n, V) = \inf_{z \in V} \|x_n - z\|_X.$$

By the definition of the infimum, there exists $z_n \in V$ such that

$$d_n \leq \|x_n - z_n\|_X \leq \left(1 + \frac{1}{n}\right) d_n. \quad (6.2)$$

In addition,

$$(I - A)(x_n - z_n) = y_n.$$

If (d_n) is bounded, then $(x_n - z_n)$ is bounded and hence, as previos, changing x_n to $x_n - z_n$, we obtain that $y \in \text{Im}(I - A)$.

Let us show that (d_n) cannot be unbounded. If it is, then there exists a subsequence, also denoted by (d_n) , such that

$$d_n \rightarrow \infty, \quad n \rightarrow \infty.$$

We consider

$$u_n = \frac{x_n - z_n}{\|x_n - z_n\|_X}.$$

Then

$$\forall n \in \mathbb{N} \quad \|u_n\|_X = 1 \text{ and } (I - A)u_n = \frac{y_n}{\|x_n - z_n\|_X} \rightarrow 0, \quad n \rightarrow \infty,$$

since

$$\left\| \frac{y_n}{\|x_n - z_n\|_X} \right\|_X = \frac{\|y_n\|_X}{\|x_n - z_n\|_X} \leq \frac{\sup_n \|y_n\|_X}{d_n} \rightarrow 0.$$

Therefore, by the previos argument, it follows that there exists a subsequence (u_{n_k}) such that $u_{n_k} \rightarrow u$ for a $u \in V$. In other hand,

$$x_{n_k} - z_{n_k} - \|x_{n_k} - z_{n_k}\|_X u = (u_{n_k} - u)\|x_{n_k} - z_{n_k}\|_X,$$

and in addition

$$z_{n_k} + \|x_{n_k} - z_{n_k}\|_X u \in V.$$

Consequently, thanks to inequality (6.2), we have

$$\begin{aligned} \|u_{n_k} - u\|_X \left(1 + \frac{1}{n_k}\right) d_{n_k} &\geq \|(u_{n_k} - u)\|x_{n_k} - z_{n_k}\|_X\|_X \\ &= \|x_{n_k} - (z_{n_k} + \|x_{n_k} - z_{n_k}\|_X u)\|_X \geq d_{n_k}, \end{aligned}$$

from where

$$\|u_{n_k} - u\|_X \geq \frac{n_k}{n_k + 1}, \quad \lim_{n_k \rightarrow \infty} \frac{n_k}{n_k + 1} = 1,$$

but this is a contradiction with $\|u_{n_k} - u\| \rightarrow 0$ for $n_k \rightarrow \infty$. Hence, (d_n) is bounded and the boundeness of $\text{Im}(I - A)$ is proved.

□

6.3 Strong and weak continuity of the linear operators in a Hilbert space

Let H be a Hilbert space and $A : H \rightarrow H$ be a linear operator on H .

Let us recall that if A is continuous (bounded) operator on H , then

$$x_n \rightarrow x \text{ in } H \implies Ax_n \rightarrow Ax \text{ in } H,$$

i.e. every strongly convergent sequence in H a bounded (or continuous) operator maps to a strongly convergent sequence in H .

Let us give now a very important theorem for the weak convergent sequences:

Theorem 6.3.1. *Let H be a Hilbert space, $A : H \rightarrow H$ be a linear operator on H and a sequence $(x_k)_{k \in \mathbb{N}} \subset H$ converge weakly to x in H .*

Then

1. *if A is continuous, then $Ax_k \rightharpoonup Ax$ for $k \rightarrow +\infty$ in H (a continuous operator preserves the weak convergence and can be called also weakly continuous);*
2. *if A is compact, then $Ax_k \rightarrow Ax$ for $k \rightarrow +\infty$ in H (a compact operator maps a weak convergent sequence to a strongly convergent sequence).*

Proof

The proof is given for the completeness, but it can be omitted.

1. Using the linearity, it is sufficient to consider the case $x = 0$. Since $A \in \mathcal{L}(H, H)$, then there exists $A^* \in \mathcal{L}(H, H)$. Therefore, if $x_k \rightharpoonup 0$ in H , then

$$\forall y \in H, \quad \langle Ax_k, y \rangle = \langle x_k, A^*y \rangle \rightarrow 0.$$

2. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence converging weakly to 0. Thus it is bounded in H .

Let us suppose that the statement is false:

$$\|Ax_k\|_H \not\rightarrow 0 \quad \text{for } k \rightarrow +\infty.$$

Thus, there exists $\epsilon > 0$ and a subsequence x_{k_j} such that

$$\|Ax_{k_j}\| \geq \epsilon.$$

As (x_{k_j}) is a bounded set in H , the compactness of A implies that there exist $y \in H$ and a subsequence (x_{k_m}) of the sequence (x_{k_j}) such that

$$\|Ax_{k_m} - y\| \rightarrow 0.$$

Since $x_{k_m} \rightharpoonup 0$ and A is continuous, then the first point implies that $y = 0$, and hence

$$\lim_{m \rightarrow \infty} \|Ax_{k_m}\| = 0,$$

which is a contradiction.

□

Remark: The second point of Theorem 6.3.1 holds “iff” and is often used as criteria: assume that $A : H \rightarrow H$ is a linear operator on a Hilbert space H (or more generally, reflexive Banach space, for the definition see Chapter 7); then

A is a compact operator if and only if it maps weak converging sequences to
strong converging sequences on H .

This criteria holds also for linear operators mapping from one Hilbert space to an other: $A : H_1 \rightarrow H_2$, where H_1 and H_2 are two Hilbert spaces, then A is a compact operator if and only if it maps weak converging sequences in H_1 to strong converging sequences on H_2 . One of typical examples is the trace operator $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega, \mu)$, see Theorem 6.5.1.

6.4 Spectral properties of compact operators in a Hilbert space

Definition 6.4.1. Let A be a linear operator mapping a topological linear space X into itself. Then a number λ is called an **eigenvalue** of A if the equation $Ax = \lambda x$ has at least one nonzero solution, and every such solution x is called an **eigenfunction** (or **eigenvector**) of A (corresponding to the eigenvalue λ).

Lemma 6.4.1. Let H be a Hilbert space and $A : H \rightarrow H$, $A \in \mathcal{L}(H, H)$ be a self-adjoint operator in H . Then

1. all eigenvalues of A are real.
2. If x and y are eigenvectors of A corresponding to the eigenvalues λ and η respectively, such that $\lambda \neq \eta$, then x is orthogonal to y in H .

Proof

1. Let $x \neq 0$ be a eigenvector of A corresponding to λ :

$$Ax = \lambda x.$$

It implies that

$$(Ax, x)_H = \lambda(x, x)_H.$$

As A is self-adjoint, its quadratic form $(Ax, x)_H$ is real. In addition, $(x, x)_H$ is real too and $(x, x)_H \neq 0$. Therefore, $\lambda = \frac{(Ax, x)_H}{(x, x)_H}$ is a real number.

2. We have $Ax = \lambda x$ and $Ay = \eta y$. Thus

$$(Ax, y)_H = \lambda(x, y)_H, \quad (x, Ay)_H = (x, \eta y)_H.$$

Since η is real, $(x, Ay)_H = \eta(x, y)_H$, and we also have $(Ax, y)_H = (x, Ay)_H$, since A is self-adjoint. Consequently, $(\lambda - \eta)(x, y)_H = 0$ which implies, since $\lambda \neq \eta$, that $(x, y)_H = 0$, which means that $x \perp y$ in H .

□

Theorem 6.4.1. (Hilbert-Schmidt) Let $H \neq \{0\}$ be a Hilbert space and $A : H \rightarrow H$ be a compact self-adjoint operator in H . Then there is an orthonormal system $\phi_1, \phi_2, \dots$ of eigenvectors of A which forms an orthonormal basis in H . In addition,

1. All nonzero eigenvalues $\lambda \neq 0$ of A have a finite multiplicity.
2. The set of different eigenvalues of A is finite or countable.
3. If the set of eigenvalues of A is countable, then the eigenvalues form a sequence (λ_n) which converges toward 0.

Remark: Let $H \neq \{0\}$ be a Hilbert space and $A : H \rightarrow H$ be a compact self-adjoint operator in H . Then, by Hilbert-Schmidt theorem, there is an orthonormal system $\phi_1, \phi_2, \dots$ of eigenvectors of A which forms an orthonormal basis in H . In particular, all $u \in H$ can be uniquely presented as

$$u = \sum_{\lambda_i \neq 0} \langle \phi_i, u \rangle \phi_i + \psi,$$

where $\psi \in \text{Ker } A$. If $\text{Ker } A = \{0\}$, then

$$\forall u \in H \quad u = \sum_i \langle \phi_i, u \rangle \phi_i \quad \text{and} \quad Au = \sum_i \lambda_i \langle \phi_i, u \rangle \phi_i.$$

6.5 Compactness of the injection of $H^1(\Omega)$ to $L^2(\Omega)$ and of the trace operator

Definition 6.5.1. The operator $i_{L^2(\Omega)} : H^1(\Omega) \rightarrow L^2(\Omega)$ is called **the injection operator of $H^1(\Omega)$ to $L^2(\Omega)$** , if it takes an element $u \in H^1(\Omega)$ and associates to it $i_{L^2(\Omega)}(u) = u \in L^2(\Omega)$, i.e. the same u , but viewed this time as an element of $L^2(\Omega)$.

Remark: Obviously, $i_{L^2(\Omega)}$ is linear. It is also easy to see that it is bounded or continuous:

$$\|i_{L^2(\Omega)}(u)\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}.$$

To complete the application of the introduced theory of compact operators, our main goal now is to understand the meaning of the following theorem:

Theorem 6.5.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a compact Lipschitz or self-similar d -set boundary $\partial\Omega$ ($n - 1 \leq d < n$) with a d -measure μ . Then

1. the injection operator $i_{L^2(\Omega)} : H^1(\Omega) \rightarrow L^2(\Omega)$ is linear compact operator;

2. the trace operator $\text{Tr} : H^1(\Omega) \rightarrow L^2(\Gamma, \mu)$ for a compact $\Gamma \subset \partial\Omega$ is a linear compact operator.

Attention: Theorem 6.5.1 contains very crucial results, absolutely to know. In the framework of this course the theorem is admitted without proof (the main ideas of the proof could be found in [5]).

One of the main corollaries of this Theorem is due to Point 2 of Theorem 6.3.1 and the weak compactness of bounded sets in the Hilbert spaces:

Corollary 6.5.1. *For any bounded sequence $(u_k)_{k \in \mathbb{N}}$ in $H^1(\Omega)$, i.e. for which there exists a constant $C > 0$ such that $\|u_k\|_{H^1(\Omega)} \leq C$ for all $k \in \mathbb{N}$, there exists a weakly convergent (to u) in $H^1(\Omega)$ subsequence $(u_{k_j})_{j \in \mathbb{N}}$, which converges strongly in $L^2(\Omega)$*

$$u_{k_j} \rightarrow u \quad \text{in } L^2(\Omega), \text{ for } j \rightarrow +\infty$$

and for which the corresponding sequence of traces on Γ converge strongly in $L^2(\Gamma, \mu)$

$$\text{Tr} u_{k_j} \rightarrow \text{Tr} u \quad \text{in } L^2(\Gamma, \mu), \text{ for } j \rightarrow +\infty.$$

Let us also notice that for any proper Hilbert subspace of $H^1(\Omega)$, for instance $V(\Omega)$ or $H_0^1(\Omega)$ the corresponding injection and trace operators are also compact.

Problem 6.5.1. *Justify and explain why $i_{L^2(\Omega)} : V(\Omega) \rightarrow L^2(\Omega)$ and $\text{Tr} : V(\Omega) \rightarrow L^2(\Gamma, \mu)$ are also compact.*

Chapter 7

Parametric optimization

Let us start by introducing the notion of reflective spaces.

7.1 Reflexivity

Definition 7.1.1. The **bidual** space of X is noted by X^{**} and defined by

$$X^{**} := (X^*)^* = \mathcal{L}(X^*, \mathbb{R}) = \mathcal{L}(\mathcal{L}(X, \mathbb{R}), \mathbb{R}).$$

It is the normed space of a linear continuous functional from X^* to $\mathbb{R}$ with the norm:

$$\|F\|_{\mathcal{L}(X^*, \mathbb{R})} = \sup_{f \in X^*, \|f\|_{\mathcal{L}(X, \mathbb{R})} \leq 1} |\langle F, f \rangle|.$$

Let us fix $x \in X$. Then for $f \in X^*$ the mapping $F_x : X^* \rightarrow \mathbb{R}$ such that $f \mapsto \langle f, x \rangle$ is a linear continuous functional on X^* .

There is a natural injection Φ from X to X^{**} . Given $x \in X$, associate $\Phi(x) = F_x \in X^{**}$ defined by a linear continuous functional on X^* : $F_x : f \in X^* \mapsto \langle f, x \rangle \in \mathbb{R}$. Thus we have the equality:

$$\langle F_x, f \rangle_{X^{**}, X^*} = \langle f, x \rangle_{X^*, X} \quad \forall x \in X, \quad \forall f \in X^*.$$

The notation $\langle x, y \rangle_{X, Y}$ means that $x \in X, y \in Y$ and $\langle x, y \rangle = x(y)$.

Proposition 7.1.1. Let X be a normed space. Then the natural injection $\Phi : X \rightarrow X^{**}$ is a linear isometric (thus continuous) operator:

$$\|F_x\|_{X^{**}} = \|x\|_X.$$

As Φ is isometric, it means that $X \approx \Phi(X) \subset X^{**}$, where $\Phi(X)$ is a subspace of X^{**} . If Φ is surjective (and then bijective), X and X^{**} can be identified.

Definition 7.1.2. We say that X is **reflexive** if $\Phi(X) = X^{**}$. In this case we write $X = X^{**}$ understanding the isometric equivalence.

Example 7.1.1. 1. $\mathbb{R}^n$ is a reflexive space.

2. ℓ^p and L^p for $1 < p < \infty$ are reflexive spaces (by the Holder inequality, $(L^p)^* = L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$, see [15] for the proof).

3. Let c_0 be the space of all sequences $x = (x_1, \dots, x_k, \dots)$ converging to zero, with the norm

$$\|x\| = \sup_k |x_k|.$$

Then the space c_0^* is isomorphic to the space ℓ^1 of all absolutely summable sequences. So $c_0^* = \ell^1$, and $(\ell^1)^* = \ell^\infty$. Therefore, $c_0 \neq c_0^{**}$.

4. L^∞ and L^1 are not reflexive spaces (see [15] for the proof). We notice that $(L^1)^* = L^\infty$ in the sense of isomorphism, but $(L^\infty)^* \neq L^1$ (it can be only associated with a proper subset of L^1 [15]).

7.2 Compacts in non reflexive spaces

We have seen in Chapter 6 that in an infinite dimensional normed space $(X, \|\cdot\|_X)$ a closed non trivial ball $\overline{B_r}(0) = \{x \in X \mid \|x\|_X \leq r\}$ is not compact (strongly). But if $(X, \|\cdot\|_X)$ is a Hilbert space, then $\overline{B_r}(0)$ is weakly compact (corresponding to the weak convergence).

All Hilbert spaces are examples of reflexive normed vector spaces, *i.e.* the spaces which are isomorph to their bi-dual spaces: $X \approx X^{**}$. If X is a Hilbert space, by the Riesz representation Theorem $X^* = \mathcal{L}(X, \mathbb{C}) \approx X$, and hence $X^{**} \approx X$.

Without going to more details of the functional analysis, you need just to know that

Attention: If $(X, \|\cdot\|_X)$ is not reflexive, then $X \subsetneq X^{**}$ and $\overline{B_r}(0)$ is not weakly compact.

This is the problem, because for the parametric optimization we need to work with closed balls of $L^\infty(\Omega)$, which is not reflexive (see [15]).

Let $X = L^1(\Omega)$, then $L^\infty(\Omega)$ is its dual space ($X^* = L^\infty(\Omega)$). On this dual space X^* , which is a Banach space (independently on the nature of the space X), it is possible to define three types of convergence for a sequence $(x_n) \subset X^*$:

1. the strong convergence: $(x_n) \subset X^*$ converges strongly to $x \in X^*$ ($x_n \rightarrow x$) iff $\|x_n - x\|_{X^*} \rightarrow 0$ for $n \rightarrow +\infty$.
2. the weak convergence: $(x_n) \subset X^*$ converges weakly to $x \in X^*$ ($x_n \rightharpoonup x$) iff for all $f \in X^{**}$ $f(x_n) := \langle f, x_n \rangle_{X^{**}, X^*} \rightarrow \langle f, x \rangle_{X^{**}, X^*} =: f(x)$ for $n \rightarrow +\infty$.
3. the weak* convergence: $(x_n) \subset X^*$ converges weakly* to $x \in X^*$ ($x_n \xrightarrow{*} x$) iff for all $y \in X$ $x_n(y) := \langle x_n, y \rangle_{X^*, X} \rightarrow \langle x, y \rangle_{X^*, X} =: x(y)$ for $n \rightarrow +\infty$.

Remark:

1. If X is not reflexive, as $L^\infty(\Omega)$, then

$$\boxed{\rightarrow \Rightarrow \rightharpoonup \Rightarrow \overset{*}{\rightharpoonup}, \quad \text{but not converse!}}$$

2. If $X^{**} = X$, then $\rightarrow \equiv \overset{*}{\rightharpoonup}$ (this is the case of Hilbert spaces).
 3. If $\dim X < \infty$ then $\rightarrow \equiv \rightharpoonup \equiv \overset{*}{\rightharpoonup}$.

Actually all properties of the weak convergence studied in Chapter 6 hold for the weak* convergence.

But the most important fact for us here is Theorem of Banach-Alaoglu-Bourbaki:

Theorem 7.2.1. (Banach-Alaoglu-Bourbaki) Let $(X, \|\cdot\|_X)$ be a vector normed space and X^* be its dual space. Then the closed ball $B_{X^*} = \{f \in X^* \mid \|f\|_{X^*} \leq 1\}$ of X^* is weakly* compact.

Let us go back to our space $L^\infty(\Omega)$. As it was noticed it is the dual space of $L^1(\Omega)$. Thus we appropriate the abstract weakly* convergence to the case of L^∞ :

Definition 7.2.1. Let $(u_n)_{n \in \mathbb{N}} \subset L^\infty(\Omega)$. The sequence weakly* converges to $u \in L^\infty(\Omega)$ if

$$\forall v \in L^1(\Omega) \quad \int_{\Omega} u_n v dx \rightarrow \int_{\Omega} u v dx, \quad (7.1)$$

and in this case we denote $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$.

Besides, by Theorem of Banach-Alaoglu-Bourbaki

$$B_{L^\infty(\Omega)} = \{u \in L^\infty(\Omega) \mid \|u\|_{L^\infty(\Omega)} \leq 1\}$$

is weakly* compact in $L^\infty(\Omega)$.

In what follows we also need the following proposition:

Proposition 7.2.1. If $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ (i.e. in X^*) and $v_n \rightarrow v$ in $L^1(\Omega)$ (i.e. in X), then

$$\int_{\Omega} u_n v_n dx \rightarrow \int_{\Omega} u v dx.$$

Proof

It is sufficient to write

$$\begin{aligned} \left| \int_{\Omega} u_n v_n dx - \int_{\Omega} u v dx \right| &\leq \left| \int_{\Omega} u_n (v_n - v) dx \right| + \left| \int_{\Omega} (u_n - u) v dx \right| \\ &\leq \|u_n\|_{L^\infty(\Omega)} \|v_n - v\|_{L^1(\Omega)} + \left| \int_{\Omega} u_n v dx - \int_{\Omega} u v dx \right|. \end{aligned}$$

We notice that as $(u_n)_{n \in \mathbb{N}}$ converges weakly* in $L^\infty(\Omega)$, then it is bounded:

$$\exists M > 0 : \forall n \in \mathbb{N} \quad \|u_n\|_{L^\infty(\Omega)} \leq M,$$

and the last term $\left| \int_{\Omega} u_n v dx - \int_{\Omega} u v dx \right| \rightarrow 0$ by the definition of the weak* convergence in $L^\infty(\Omega)$. In addition, the strong convergence $v_n \rightarrow v$ in $L^1(\Omega)$ ensures that $\|v_n - v\|_{L^1(\Omega)} \rightarrow 0$ for $n \rightarrow +\infty$. $\square$

7.3 Parametric optimization problem

We are interested in finding an optimal distribution of the porous material in a reflexive wall of a given shape, allowing to minimize the acoustical energy. More precisely, we consider that the wall can be made of two materials, one is an efficient absorbent, and the other is a reflective material. The shape of the wall is fixed and the quantity of the porous material too. Usually, the reflective materials are cheaper than the absorbent ones. Hence, one of the problems of interest is to perform at least the same energy absorption with the optimized configuration of a small quantity of porous material as in the case when all wall is done from the absorbent material. In many applications to have all wall from the porous material is not possible by the construction (at liners of the plan's reactors) or is not acceptable by aesthetic design questions (perforated wood panels with a fibrous material inside in offices are more convenient as a panel of fibrous material). Thus, for the perforated panels, there is a question where and of which size it is more convenient to make the holes filled with the porous material to absorb the noise maximally.

We can model this situation for instance in the application of perforated panels in the concert room (see Fig. 3.7):

$$\begin{cases} (\Delta + k^2)u = 0 \text{ on } \Omega; \\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{Neu}; \\ u = g \text{ on } \Gamma_{Dir}; \\ \frac{\partial u}{\partial n} + \alpha\chi u = 0 \text{ on } \Gamma, \end{cases} \quad (7.2)$$

where $k > 0$ is a fixed wavenumber and g is a fixed source of the noise (typically, g can be a function of k , and α too). Here χ is a characteristic function of holes filled with the porous material:

$$\forall x \in \Gamma \quad \chi(x) = \begin{cases} 1, & \text{there is a porous material in } x \\ 0, & \text{no porous material in } x. \end{cases} \quad (7.3)$$

The sole and the ceiling are modeled by the homogeneous Neumann boundary condition and are supposed to be reflective. The noise source is projected to the boundary Γ_{Dir} and modeled by the Dirichlet boundary condition with the function g . The perforated wall is the boundary part Γ , and it has reflective parts when $\chi = 0$ and the inclusion of a porous material if $\chi = 1$. Indeed, if $x \in \Gamma$ for which $\chi(x) = 0$, then in this point on Γ it is imposed the homogeneous Neumann boundary condition, *i.e.* no absorption, the wave is totally reflected. But if $\chi(x) = 1$, then in this point of Γ we have the Robin boundary condition

$$\frac{\partial u}{\partial n} + \alpha u = 0$$

with $\text{Re}(\alpha) > 0$ and $\text{Im}(\alpha) < 0$, ensuring a partial absorption and a partial reflection as explained in Chapter 3.

7.4 Existence of an optimal shape

7.4.1 Some generalities on an optimization problem

Definition 7.4.1. Let X be a Banach space and $K \subset X$ be a non empty subset. For a functional (a criteria) $J : X \rightarrow \mathbb{R}$ we consider $\inf_{v \in K \subset X} J(v)$. A **minimizing sequence** of criteria J on the set K is called a sequence $(u_n)_{n \in \mathbb{N}}$ such that

$$u_n \in K \quad \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow +\infty} J(u_n) = \inf_{v \in K} J(v).$$

Attention: By the definition of the infimum of J on K the minimizing sequence exists all times!

The question of the optimization problem is the following:

Question: Does there exist $u \in K$ such that $J(u) = \min_{v \in K} J(v)$?

To solve this question we use the following well-knowing fact:

Theorem 7.4.1. *A continuous function on a compact takes its minimum and maximum values on it (i.e. its supremum and infimum on a compact are realized on it).*

So in what follows we need to care about

1. the continuity of J (as a function of v);
2. the property of K to be compact.

7.4.2 Existence of an optimal distribution χ minimizing the acoustical energy

We fix a wavenumber $k > 0$ and, hence, don't care about the dependance on k of α , g and u .

The acoustical energy of the Helmholtz problem (7.2) is given by

$$J(\chi, u(\chi)) = \int_{\Omega} |u(\chi)|^2 dx, \quad (7.4)$$

where u is the weak solution of the Helmholtz problem (7.2).

If we change the distribution of the porous material χ , it will change u and then the energy J . Thus u and J are functions of χ (see Fig. 7.1 for two different examples of χ).

Thus, we need to define our K , the set on which we want to minimize J , i.e. the set of all χ which are of interest for us (the elements v of K in the notations of the previous Section this time are characteristic functions χ). In the shape optimization framework, this set is usually called the set of **admissible shapes**, and it contains all possible constraints of the constructions and all information about the desired optimal shape what we are looking for.

Remark: More generally, instead of the functional J defined by L^2 -norm, it is possible to minimize any functional defined by an equivalent norm to the canonical H^1 -norm of u (see [54]). We don't do it here for the simplicity of the further computations.

For example, it is practical to fix the volume fraction β of the absorbing material of the wall (a number between 0 and 1, if $\mu(\Gamma) = 1$):

$$0 < \int_{\Gamma} \chi d\mu = \beta < \mu(\Gamma) = \int_{\Gamma} 1 d\mu = 1.$$

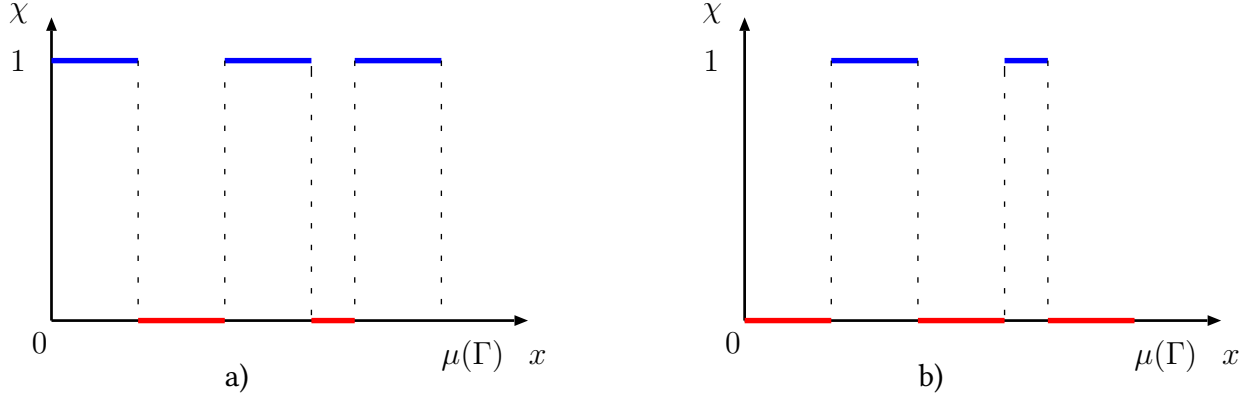


Figure 7.1 – Examples of two different distributions of the porous material given by χ (in an assumption, for instance, of the two dimensional model with $x \in \Gamma$ moving from its one end to on other, which actually could be viewed with the help of the natural parametrization of Γ , *i.e.* the parametrization which associates any point of Γ with a length of the part of Γ starting from the chosen one of two ends of Γ and finishing by the current point x). Here $\mu(\Gamma)$ is the length of Γ (supposed to be finite). The red lines correspond to the absence of the porous material, the blue lines correspond to the holes filled with the porous material. On the case a) we have three holes and on the case b) we have two holes.

The two limit cases $\beta = 0$ and $\beta = 1$ are excluded.

Therefore, assuming $\mu(\Gamma) = 1$, we define the space of admissible χ :

$$U_{ad}(\beta) = \{\chi \in L^\infty(\Gamma, \mu) \mid \forall x \in \Gamma \quad \chi(x) \in \{0, 1\}, \quad 0 < \beta = \int_\Gamma \chi d\mu < 1\}. \quad (7.5)$$

The set $U_{ad}(\beta)$ is thus a subset of $L^\infty(\Gamma, \mu)$ consisting of all functions taking only two values 0 or 1 on Γ and define a fixed value of β in $]0, 1[$.

Thus the optimization problem, for a fixed frequency $k > 0$ and the porous material, which determine the complexe parameter α , reads:

Definition 7.4.2. (Parametric optimization problem) For a fixed $\beta > 0$ and the source of the noise g to find $\chi_{opt} \in U_{ad}(\beta)$ for which there exists the solution $u(\chi_{opt})$ of the Helmholtz problem (7.2) considered with $\chi = \chi_{opt}$, such that

$$J(\chi_{opt}, u(\chi_{opt})) = \min_{\chi \in U_{ad}(\beta)} J(\chi, u(\chi)).$$

The main problem is that the set of the admissible shapes $U_{ad}(\beta)$ is not closed for the weak* convergence of $L^\infty(\Gamma, \mu)$ [41]: if a sequence of characteristic functions $(\chi_j)_{j \in \mathbb{N}}$ converges weakly* in $L^\infty(\Gamma, \mu)$ to a function $h \in L^\infty(\Gamma, \mu)$, it does not follows and it is not possible to ensure (there are counter-examples) that the weak* limit function h is a characteristic function, *i.e.* takes only two values 0 and 1.

Remark: To ensure that the limit of a sequence of characteristic functions is a characteristic function, we need the strong convergence in $L^\infty(\Gamma, \mu)$. Here, we have to work with the weak* convergence, which does not imply the strong one.

Thus, $U_{ad}(\beta)$ is not a weak* compact. Consequently, the parametric shape optimization problem defined in 7.4.2 cannot be generally solved on this $U_{ad}(\beta)$.

But it can be solved on an other set, containing $U_{ad}(\beta)$, this time weak* compact in $L^\infty(\Gamma, \mu)$ in the way that $\inf_{\chi \in U_{ad}(\beta)} J(\chi, u(\chi))$ is given by the minimum of J on this new compact class. This method is called **the relaxation method**.

7.4.3 Relaxation method

We recall that for the simplicity we have taken $\mu(\Gamma) = 1$. Instead of solving the optimization problem on $U_{ad}(\beta)$ we will solve it on its (convex) closure:

$$U_{ad}^*(\beta) = \{\chi \in L^\infty(\Gamma, \mu) \mid \forall x \in \Gamma \quad \chi(x) \in [0, 1], \quad 0 < \int_\Gamma \chi d\mu = \beta < 1\}. \quad (7.6)$$

In this case it is known (for the ideas of the proof see [41, Proposition 7.2.14])

Theorem 7.4.2. *Let $\beta \in]0, 1[$ be fixed. If $U_{ad}^*(\beta)$ is given by (7.6), then*

1. $U_{ad}(\beta) \subset U_{ad}^*(\beta)$,
2. $U_{ad}^*(\beta) = \overline{U_{ad}(\beta)}^*$
 $= \{\text{the smallest set closed by the convergence } \xrightarrow{*} \text{ in } L^\infty(\Gamma, \mu) \text{ which contains } U_{ad}(\beta)\}.$
3. $U_{ad}^*(\beta)$ is convexe.
4. $U_{ad}^*(\beta)$ is compact in $L^\infty(\Gamma, \mu)$ for the weak* convergence.

Therefore, we denote by $J^* : U_{ad}^*(\beta) \rightarrow \mathbb{R}$ the functional

$$\forall \chi \in U_{ad}^*(\beta) \quad J^*(\chi, u(\chi)) = \int_\Omega |u(\chi)|^2 dx,$$

which in addition satisfies

$$J^*(\chi, u(\chi))|_{U_{ad}} = J(\chi, u(\chi)) = \int_\Omega |u(\chi)|^2 dx.$$

Here $u(\chi)$ is the weak solution of system (7.2) found for a chosen χ .

In what follows, for the simplicity of the mathematical studies, we consider instead of (7.2) the following system

$$\begin{cases} (\Delta + k^2)u = f \text{ on } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{Neu}, \\ u = 0 \text{ on } \Gamma_{Dir}, \\ \frac{\partial u}{\partial n} + \alpha \chi u = \eta \text{ on } \Gamma \end{cases} \quad (7.7)$$

with $\eta \in L^2(\Gamma, \mu)$ and $f \in L^2(\Omega)$.

Actually, by the linearity of the problem, knowing g on Γ_{Dir} , it is possible to find η and f in the aim to have the homogeneous Dirichlet condition on Γ_{Dir} (see the next Remark), and also conversely, to find a suitable g on Γ_{Dir} making $\eta = f = 0$. Hence, we work with (7.7), which allows us to have less complicated theoretical mathematical treatment.

Remark: To solve system (7.2) with $g \neq 0$ we write $u = \hat{u} + \hat{g}$, where $\hat{g} \in H^1(\Omega)$ with $\text{Tr}\hat{g}|_{\Gamma_{Dir}} = g$ in the aim to have $\text{Tr}\hat{u}|_{\Gamma_{Dir}} = 0$. From (7.2) we obtain

$$\begin{aligned} (\Delta + k^2)\hat{u} &= -(\Delta + k^2)\hat{g} \text{ on } \Omega, \\ \frac{\partial \hat{u}}{\partial n} &= -\frac{\partial \hat{g}}{\partial n} \text{ on } \Gamma_{Neu}, \\ \hat{u} &= 0 \text{ on } \Gamma_{Dir}, \\ \frac{\partial \hat{u}}{\partial n} + \alpha\chi\hat{u} &= -\left(\frac{\partial \hat{g}}{\partial n} + \alpha\chi\hat{g}\right) \text{ on } \Gamma. \end{aligned}$$

To simplify the obtained system on $\hat{u}$, it is more practical to take as $\hat{g}$ (it is actually an extension of g and it can be taken as we prefer just keeping true the assumptions that it is sufficient regular with $\text{Tr}\hat{g}|_{\Gamma_{Dir}} = g$) the weak solution of the following system:

$$\begin{aligned} \Delta \hat{g} &= 0 \text{ on } \Omega, \\ \frac{\partial \hat{g}}{\partial n} &= 0 \text{ on } \Gamma_{Neu} \cup \Gamma, \\ \text{Tr}\hat{g} &= g \text{ on } \Gamma_{Dir}. \end{aligned}$$

With this choice of $\hat{g}$, the system for $\hat{u}$ becomes

$$\begin{aligned} (\Delta + k^2)\hat{u} &= -k^2\hat{g} \text{ on } \Omega, \\ \frac{\partial \hat{u}}{\partial n} &= 0 \text{ on } \Gamma_{Neu}, \\ \hat{u} &= 0 \text{ on } \Gamma_{Dir}, \\ \frac{\partial \hat{u}}{\partial n} + \alpha\chi\hat{u} &= -\alpha\chi\hat{g} \text{ on } \Gamma. \end{aligned}$$

It means that we need to solve a system of the same type as (3.15) (assuming $g = 0$ in (3.15)) with

$$f := -k^2\hat{g}, \quad \text{Tr}h := -\alpha\chi\text{Tr}\hat{g}$$

(in the new notations of system (7.7) $\eta = -\alpha\chi\text{Tr}\hat{g}$ on Γ) and thus obtain by Theorem 3.2.1 the unique weak solution $\hat{u} \in V(\Omega)$.

Attention: To solve (3.15) with $g \neq 0$, it means to find $u \in H^1(\Omega)$ such that $\text{Tr}(u - \hat{g}) = 0$ on Γ_{Dir} . It means that for an extension of g from Γ_{Dir} to $\hat{g}$ on Ω ($\hat{g} \in H^1(\Omega)$) we have $u - \hat{g} \in V(\Omega)$.

Attention: In (3.15), it was defined $h \in H^1(\Omega)$, and then its trace is taken on Γ . It was done in the future idea that Γ could be a moving boundary of the shape optimization problem formulated in Chapter 5.4 and thus, it is more natural to define h (and α) in an area where Γ could change and then take there traces on it.

Here, in the parametric optimization, the situation is different: the shape of the boundary is kept fixed, but the distribution of the porous material, given by χ , changes. Thus we are not interested to know h and α in an area, but only on our fixed shape of Γ . Thus, we define η , knowing only on Γ without a requirement that η is a trace of an element of $H^1(\Omega)$, but just taking it in $L^2(\Gamma, \mu)$.

However, we know that any trace of an element of $H^1(\Omega)$ belongs to $L^2(\Gamma, \mu)$! Actually, $B(\Gamma) \subset L^2(\Gamma, \mu)$.

Let us recall that the weak solution of system (7.7) $u \in V(\Omega)$ is understood in the following variational sense

$$\begin{aligned} \forall \phi \in V(\Omega) \quad \int_{\Omega} \nabla u \nabla \bar{\phi} dx - k^2 \int_{\Omega} u \bar{\phi} dx + \int_{\Gamma} \operatorname{Re}(\alpha) \chi \operatorname{Tr} u \operatorname{Tr} \bar{\phi} d\mu \\ + i \int_{\Gamma} \operatorname{Im}(\alpha) \chi \operatorname{Tr} u \operatorname{Tr} \bar{\phi} d\mu = \int_{\Gamma} \eta \operatorname{Tr} \bar{\phi} d\mu - \int_{\Omega} f \bar{\phi} dx. \end{aligned} \quad (7.8)$$

Thus, we have an analogue of Theorem 3.2.1 (the adaption is straightforward and can be admitted without details):

Theorem 7.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz or self-similar d -set boundary ($n - 1 < d < n$) with a d -measure μ . Assume*

$$\partial\Omega = \Gamma_{Dir} \cup \Gamma_{Neu} \cup \Gamma$$

such that

$$\mu(\Gamma_{Neu} \cap \Gamma_{Dir}) = \mu(\Gamma_{Neu} \cap \Gamma) = \mu(\Gamma_{Dir} \cap \Gamma) = 0. \quad (7.9)$$

Let Γ_{Dir} and Γ be also closed subsets of $\partial\Omega$ and d -sets with the same properties as $\partial\Omega$ itself ($\mu(\Gamma) > 0$ and $\mu(\Gamma_{Dir}) > 0$). Let in addition

$$\operatorname{Re}(\alpha(x)) > 0, \quad \operatorname{Im}(\alpha(x)) < 0$$

be smooth functions (at least continuous) on Γ . Let $\chi \in L^\infty(\Gamma, \mu)$ be a nonnegative on Γ and positive with a positive minimum on a subset positive μ -measure. Then for all $f \in L^2(\Omega)$, $\eta \in L^2(\Gamma, \mu)$, and $k > 0$ there exists a unique solution $u \in V(\Omega)$ of the Helmholtz problem (7.7) in the following sense: for all $v \in V(\Omega)$ it holds (7.8).

Moreover, the solution of problem (7.7) $u \in V(\Omega)$, continuously depends on the data: there exists a constant $C > 0$, depending only on χ , $\|\operatorname{Tr}\|_{\mathcal{L}(V(\Omega), L^2(\Gamma, \mu))}$, α , k and on $C_P(\Omega)$, such that

$$\|u\|_{V(\Omega), \chi} \leq C \left(\|f\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Gamma, \mu)} \right), \quad (7.10)$$

where $C_P(\Omega)$ is the Poincaré constant associated to Ω and by $\|\cdot\|_{V(\Omega), \chi}$ is denoted the equivalent to $\|\cdot\|_{H^1(\Omega)}$ norm on $V(\Omega)$ given by

$$\|u\|_{V(\Omega), \chi} = \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\sqrt{\operatorname{Re}(\alpha)\chi} \operatorname{Tr} u\|_{L^2(\Gamma, \mu)}^2 \right)^{\frac{1}{2}}. \quad (7.11)$$

Let us however notice that norm (7.11) is associated to the following inner product:

$$\forall u, v \in V(\Omega) \quad (u, v)_{V(\Omega), \chi} = \int_{\Omega} \nabla u \nabla \bar{v} dx + \int_{\Gamma} \operatorname{Re}(\alpha) \chi \operatorname{Tr} u \operatorname{Tr} \bar{v} d\mu. \quad (7.12)$$

Therefore, the variational formulation (7.8) can be rewritten as

$$\begin{aligned} \forall \phi \in V(\Omega) \quad (u, \phi)_{V(\Omega), \chi} - k^2 (u, \phi)_{L^2(\Omega)} + i(\operatorname{Im}(\alpha) \chi \operatorname{Tr} u, \operatorname{Tr} \phi)_{L^2(\Gamma, \mu)} \\ = (\eta, \operatorname{Tr} \phi)_{L^2(\Gamma, \mu)} - (f, \phi)_{L^2(\Omega)}. \end{aligned} \quad (7.13)$$

To solve the parametric optimization problem on $U_{ad}^*(\beta)$ we need to ensure that the constant C in estimate (7.10) does not depend on χ , when $\chi \in U_{ad}^*(\beta)$. It is actually the case, and it could be simply viewed as the corollary of the uniform upper boundness of the L^∞ norm of all χ from $U_{ad}^*(\beta)$. Indeed, let us notice that $\|\chi\|_{L^\infty(\Gamma, \mu)} = 1$ for all $\chi \in U_{ad}(\beta)$, while for all $\chi \in U_{ad}^*(\beta)$ it holds

$$0 < \beta \leq \|\chi\|_{L^\infty(\Gamma, \mu)} \leq 1. \quad (7.14)$$

In addition, for all $\chi \in U_{ad}^*(\beta)$ $\|\cdot\|_{V(\Omega), \chi}$ is equivalent to $\|\cdot\|_{H_0^1(\Omega)}$ with uniform on χ constants: there exist $C_0 > 0$ independent on $\chi \in U_{ad}^*(\beta)$ such that

$$\forall v \in V(\Omega) \quad \|v\|_{H_0^1(\Omega)} \leq \|v\|_{V(\Omega), \chi} \leq C_0 \|v\|_{H_0^1(\Omega)}. \quad (7.15)$$

For the proof it is sufficient to apply the continuity of the trace operator and the Poincaré inequality, and to obtain $C_0 = (1 + \|\alpha_R\|_{C(\Gamma)} C(\|\operatorname{Tr}\|) C_P(\Omega))$. Therefore, it holds

$$\|u\|_{H_0^1(\Omega)} \leq \|u\|_{V(\Omega), \chi} \leq C(\alpha, k, C_P(\Omega), \|\operatorname{Tr}\|) \left(\|f\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Gamma, \mu)} \right), \quad (7.16)$$

for a constant $C(\alpha, k, C_P(\Omega), \|\operatorname{Tr}\|) > 0$ independent on $\chi \in U_{ad}^*(\beta)$.

Consequently, we have

Lemma 7.4.1. *Let $\beta > 0$ be fixed. For all $\chi \in U_{ad}^*(\beta)$, there exists a constant $C > 0$, depending only on $\alpha, k, \|\operatorname{Tr}\|_{\mathcal{L}(V(\Omega), L^2(\Gamma, \mu))}$ and on $C_P(\Omega)$, **but not on** χ , such that estimate (7.16) hold for the corresponding solutions of (7.7) on a fixed bounded domain Ω (with all conditions of Theorem 7.4.3).*

Thanks to Theorem 7.4.3 and Lemma 7.4.1, we can prove the continuity of the functional $J^*(\chi, u(\chi))$ on $\chi \in U_{ad}^*(\beta)$:

Proposition 7.4.1. *The functional $J^*(\chi, u(\chi)) = \int_{\Omega} |u(\chi)|^2 dx$, defined for $\chi \in U_{ad}^*(\beta)$ (see (7.6)) and $u(\chi)$, the weak solution of (7.7) in the sense of Theorem 7.4.3, is continuous on χ in the following sense: if $(\chi_j)_{j \in \mathbb{N}} \subset U_{ad}^*(\beta)$ such that*

$$\chi_j \xrightarrow{*} \chi \text{ in } L^\infty(\Gamma, \mu),$$

then $\chi \in U_{ad}^(\beta)$, $u(\chi_j) \rightarrow u(\chi)$ in $V(\Omega)$ and*

$$\begin{aligned} |J^*(\chi_j, u(\chi_j)) - J^*(\chi, u(\chi))| &= \left| \int_{\Omega} |u(\chi_j)|^2 dx - \int_{\Omega} |u(\chi)|^2 dx \right| \\ &= \left| \|u(\chi_j)\|_{L^2(\Omega)}^2 - \|u(\chi)\|_{L^2(\Omega)}^2 \right| \rightarrow 0 \quad k \rightarrow +\infty. \end{aligned} \quad (7.17)$$

Proof

As $U_{ad}^*(\beta)$ is closed for the weak* convergence, the weak* limit χ of a sequence $(\chi_j)_{j \in \mathbb{N}} \subset U_{ad}^*(\beta)$ belongs to $U_{ad}^*(\beta)$.

Let us denote by u_j the solution of (7.7) found for χ_j and by u the solution of (7.7) found for χ .

Thus, the difference $v_j = u_j - u$ is the solution of the following system:

$$\begin{cases} (\Delta + k^2)v_j = 0 \text{ on } \Omega, \\ \frac{\partial v_j}{\partial n} = 0 \text{ on } \Gamma_{Neu}, \\ v_j = 0 \text{ on } \Gamma_{Dir}, \\ \frac{\partial v_j}{\partial n} + \alpha \chi_j v_j = \alpha(\chi - \chi_j)u \text{ on } \Gamma; \end{cases} \quad (7.18)$$

We apply Theorem 7.4.3 for system (7.18):

- $f = 0$,
- $\eta = \alpha(\chi - \chi_j)\text{Tr}u$. As α is a continuous function, $\chi - \chi_j$ belongs to $L^\infty(\Gamma, \mu)$ and, since $u \in V(\Omega)$, we have $\text{Tr}u \in B(\Gamma) \subset L^2(\Gamma, \mu)$, then we deduce that $\eta \in L^2(\Gamma, \mu)$.

For all $j \in \mathbb{N}$, by Theorem 7.4.3, there exists a unique solution $v_j \in V(\Omega)$.

Moreover, the sequence $(v_j)_{j \in \mathbb{N}}$ is bounded in $V(\Omega)$, i.e. there exists a constant $C > 0$ such that

$$\forall j \in \mathbb{N} \quad \|v_j\|_{V(\Omega), \chi_j} \leq C. \quad (7.19)$$

Indeed, by estimate (7.10) (with uniform on j constant) we have

$$\|v_j\|_{V(\Omega), \chi_j} \leq C \|\alpha(\chi - \chi_j)\text{Tr}u\|_{L^2(\Gamma, \mu)} \leq C \|\alpha\|_{L^\infty(\Gamma, \mu)} \|\chi - \chi_j\|_{L^\infty(\Gamma, \mu)} \|\text{Tr}u\|_{L^2(\Gamma, \mu)}.$$

By our assumptions (see the assumptions of Theorem 7.4.3) α is continuous on Γ and Γ is compact, then $\|\alpha\|_{L^\infty(\Gamma, \mu)}$ is finite and does not depend on j . As $\chi_j \xrightarrow{*} \chi$ in $L^\infty(\Gamma, \mu)$, then the sequence $(\chi_j)_{j \in \mathbb{N}}$ is bounded in $L^\infty(\Gamma, \mu)$, and consequently the sequence $(\chi - \chi_j)_{j \in \mathbb{N}}$ is also bounded in $L^\infty(\Gamma, \mu)$.

Attention: Please be aware that $\|\chi - \chi_j\|_{L^\infty(\Gamma, \mu)} \not\rightarrow 0$ for $j \rightarrow +\infty$, since we only have $\chi_j \xrightarrow{*} \chi$ in $L^\infty(\Gamma, \mu)$ and not the strong convergence $\chi_j \rightarrow \chi$ in $L^\infty(\Gamma, \mu)$.

As u is the weak solution of (7.7) associated to χ , by Theorem 7.4.3, $u \in V(\Omega)$ and thus its trace on Γ is well defined and naturally belongs to $L^2(\Gamma, \mu)$. In addition $\|\text{Tr}u\|_{L^2(\Gamma, \mu)}$ does not depend on j .

Therefore, we conclude that the sequence $(v_j)_{j \in \mathbb{N}}$ is bounded in $V(\Omega)$ and that (7.19) holds.

As $V(\Omega)$ is a Hilbert space, by Theorem 6.1.2, there exists a weakly convergent subsequence:

$$\exists (v_{j_\ell})_{\ell \in \mathbb{N}} \subset (v_j)_{j \in \mathbb{N}} \text{ and } v \in V(\Omega) : \quad v_{j_\ell} \rightharpoonup v \text{ in } V(\Omega).$$

Now we need to show that $v = 0$.

By TD4, the variational formulation (7.8) for $f = 0$ can be presented in the following operator form:

$$((Id - k^2T)u, \phi)_{V(\Omega), \chi} = (\eta, \text{Tr}\phi)_{L^2(\Gamma, \mu)}, \quad (7.20)$$

where $Id - k^2T$ is bijective operator on $V(\Omega)$ with Id the identity operator and $T : V(\Omega) \rightarrow V(\Omega)$ a compact operator.¹ By (7.20), we have for all $j \in \mathbb{N}$ and all $\phi \in V(\Omega)$ that

$$((Id - k^2T)v_{j_i}, \phi)_{V(\Omega), \chi_{j_i}} = (\alpha(\chi - \chi_{j_i})\text{Tr}u, \text{Tr}\phi)_{L^2(\Gamma, \mu)}. \quad (7.21)$$

The operator $Id - k^2T$ is linear, continuous, and bijective on $V(\Omega)$. By its continuity, for $i \rightarrow +\infty$, $(Id - k^2T)v_{j_i} \rightharpoonup (Id - k^2T)v$ in $V(\Omega)$, and hence, in the limit (7.21) becomes

$$((Id - k^2T)v, \phi)_{V(\Omega), \chi} = 0 \text{ for all } \phi \in V(\Omega). \quad (7.22)$$

By the injectivity of the operator $Id - k^2T$, relation (7.22) implies that $v = 0$.

Let us now prove that $v_{j_i} \rightarrow 0$ in $V(\Omega)$ (strongly!). We take in (7.21) $\phi = v_{j_i}$ and notice that by the compactness of the trace operator $\text{Tr} : V(\Omega) \rightarrow L^2(\Gamma, \mu)$, we have, as $v_{j_i} \rightharpoonup 0$ in $V(\Omega)$, that $\text{Tr}v_{j_i} \rightarrow 0$ in $L^2(\Gamma, \mu)$. Therefore, we find that

$$((Id - k^2T)v_{j_i}, v_{j_i})_{V(\Omega), \chi_{j_i}} = (\alpha(\chi - \chi_{j_i})\text{Tr}u, \text{Tr}v_{j_i})_{L^2(\Gamma, \mu)} \rightarrow 0 \text{ for } i \rightarrow +\infty. \quad (7.23)$$

Remark: Let us justify that $|\int_{\Gamma} \alpha(\chi - \chi_{j_\ell})\text{Tr}u\text{Tr}\overline{v_{j_\ell}} d\mu| \rightarrow 0$ for $\ell \rightarrow +\infty$. It is a corollary of Proposition 7.2.1:

- as $\chi_j \xrightarrow{*} \chi$ in $L^\infty(\Gamma, \mu)$, then $\chi - \chi_{j_\ell} \xrightarrow{*} 0$ in $L^\infty(\Gamma, \mu)$.
- by Theorem 6.5.1, the trace operator $\text{Tr} : V(\Omega) \rightarrow L^2(\Gamma, \mu)$ is compact, and thus by Point 2 of Theorem 6.3.1, as $v_{j_\ell} \rightharpoonup v$ in $V(\Omega)$ then

$$\text{Tr}v_{j_\ell} \rightarrow \text{Tr}v \text{ in } L^2(\Gamma, \mu).$$

- as $\text{Tr}u \in L^2(\Gamma, \mu)$, then $\text{Tr}u\text{Tr}\overline{v_{j_\ell}} \in L^1(\Gamma)$ for all ℓ as a product of two elements of $L^2(\Gamma, \mu)$. In addition $\text{Tr}u \in L^2(\Gamma, \mu)$ is independent of ℓ , which implies by the previous item that

$$\text{Tr}u\text{Tr}\overline{v_{j_\ell}} \rightarrow \text{Tr}u\text{Tr}\overline{v} \in L^1(\Gamma).$$

Consequently, $\left[\|v_{j_i}\|_{V(\Omega), \chi_{j_i}}^2 - k^2(Tv_{j_i}, v_{j_i})_{V(\Omega), \chi_{j_i}} \right] \rightarrow 0$ for $i \rightarrow +\infty$. By its definition the operator $T : V(\Omega) \rightarrow V(\Omega)$ is compact (see TD4), and hence, $Tv_{j_i} \rightarrow 0$ in $V(\Omega)$, which result in $(Tv_{j_i}, v_{j_i})_{V(\Omega), \chi_{j_i}} \rightarrow 0$ for $i \rightarrow +\infty$. Finally, we obtain that $\|v_{j_i}\|_{V(\Omega), \chi_{j_i}}^2 \rightarrow 0$ and $v_{j_i} \rightarrow 0$ in $V(\Omega)$, ensuring the strong convergence $v_{j_i} \rightarrow 0$ in $V(\Omega)$.

Now, any weakly convergent sequence of solutions of system (7.18) converges to 0, *i.e.* 0 is the unique accumulation (limit) point of the sequence $(v_j)_{j \in \mathbb{N}}$, which implies that the sequence $v_j \rightarrow 0$ itself in $V(\Omega)$. Actually, $v_j = u_j - u$ and hence $u_j \rightarrow u$ in $V(\Omega)$ for $j \rightarrow +\infty$.

Finally, we have obtained that from $\chi_j \xrightarrow{*} \chi$ in $L^\infty(\Gamma, \mu)$, it follows that $\chi \in U_{ad}^*(\beta)$ and $u(\chi_j) \rightarrow u(\chi)$ in $V(\Omega)$, from where we directly have (7.17), *i.e.* the continuity of J^* on $U_{ad}^*(\beta)$.

□

¹The compactness of T follows from the compactness of the trace operator (see [13], [44, Theorem 5.10], [45, Theorem 2.1]) and the embedding $H^1(\Omega) \subset L^2(\Omega)$ (the compactness holds for any bounded extension domain [6, 62, 44]).

Thus we obtain the following theorem:

Theorem 7.4.4. *Let Γ , f , η , k , α be fixed in a way that all assumptions of Theorem 7.4.3 are satisfied on a fixed bounded domain Ω of $\mathbb{R}^n$. Then for a fixed $\beta \in]0, \mu(\Gamma)[$ there exists (at least one) optimal distribution $\chi^{opt} \in U_{ad}^*(\beta)$ and the corresponding optimal solution $u(\chi^{opt}) \in V(\Omega)$ of system (7.7), such that*

$$J^*(\chi^{opt}, u(\chi^{opt})) = \min_{\chi \in U_{ad}^*(\beta)} J^*(\chi, u(\chi)) = \inf_{\chi \in U_{ad}(\beta)} J(\chi, u(\chi)).$$

Proof

By Proposition 7.4.1 the energy functional J^* is continuous corresponding to the weak* convergence of $\chi \in U_{ad}^*(\beta)$ (in addition $u(\chi)$ is also weak* continuous on $U_{ad}^*(\beta)$). As $U_{ad}^*(\beta)$ is weakly* compact (in $L^\infty(\Gamma, \mu)$) and J^* is continuous on it, then there exists a $\chi^{opt} \in U_{ad}^*(\beta)$ (and the corresponding solution $u(\chi^{opt})$) realizing its minimum on $U_{ad}^*(\beta)$. In addition, $\min_{\chi \in U_{ad}^*(\beta)} J^*(\chi, u(\chi)) = \inf_{\chi \in U_{ad}(\beta)} J(\chi, u(\chi))$ as $U_{ad}^*(\beta)$ is the closure of $U_{ad}(\beta)$ and J^* takes the same values as J on $U_{ad}(\beta)$ (see Theorem 7.4.2). $\square$

Remark: This situation to have a minimum in the closure of $U_{ad}(\beta)$ equal to the infimum on it is schematically presented on Fig. 7.2. It is also known (from the properties of the spectral problems for $-\Delta$) that there are no unicity of an optimal χ^{opt} .

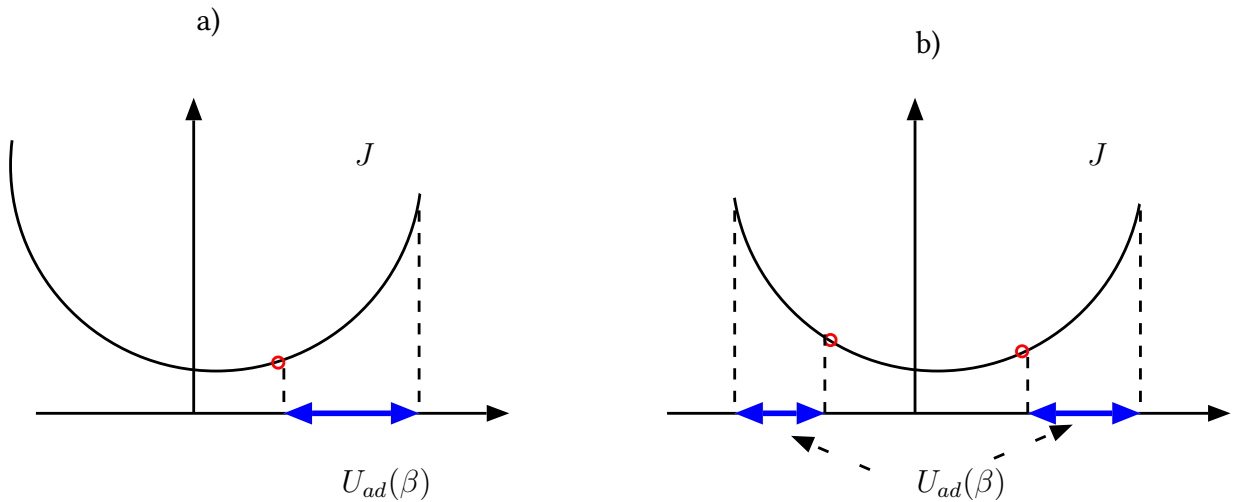


Figure 7.2 – Two schematic examples of two different choices of an open admissible set $U_{ad}(\beta)$ (in blue). Each time the minimum on $\overline{U_{ad}(\beta)}$ is realized on the point(s) corresponding to the infimum on $U_{ad}(\beta)$. The minimum values of J on $U_{ad}^*(\beta) = \overline{U_{ad}(\beta)}$ are given by the red circles, the values not reached on $U_{ad}(\beta)$. The configuration b) shows the non uniqueness of χ^{opt} giving the same minimum value of J .

7.5 Derivative of the energy

To be able to find χ^{opt} numerically, it is useful to use a numerical method based on the conjugate gradient method, *i.e.* to find a minimum of the energy J , using its derivative on χ . Thus we need to define such derivative.

7.5.1 Fréchet derivative

Definition 7.5.1 (Directional derivative). Let X be a Banach space and $F : X \rightarrow \mathbb{R}$ be a functional on X . The **directional derivative** of F in the direction h is called a linear application $h \mapsto \langle F'(x), h \rangle$ defined by

$$\langle F'(x), h \rangle := \frac{d}{d\lambda} F(x + \lambda h)|_{\lambda=0}, \quad x, h \in X, \quad \lambda \in \mathbb{R}.$$

Example 7.5.1. Let $F : C([0, 1]) \rightarrow \mathbb{R}$ such that $F(x) = \int_0^1 x(t)^2 dt$ for $x(t) \in C([0, 1])$. Let us fix $h \in C([0, 1])$ and notice that

$$\begin{aligned} F(x + \lambda h) &= \int_0^1 (x(t) + \lambda h(t))^2 dt = \\ &= \int_0^1 x(t)^2 dt + 2\lambda \int_0^1 x(t)h(t) dt + \lambda^2 \int_0^1 h^2(t) dt \end{aligned}$$

is a polynomial of the second order on λ . Thus

$$\langle F'(x), h \rangle = \frac{d}{d\lambda} F(x + \lambda h)|_{\lambda=0} = 2 \int_0^1 x(t)h(t) dt,$$

the linear application on h .

A generalization of a differentiable function on a point gives the differentiability of the operators:

Definition 7.5.2. [Fréchet derivative] Let X and Y be Banach spaces. An operator $F(x) : X \rightarrow Y$ is **differentiable (in the sense of Fréchet)** at $x \in X$, if there exists a continuous linear operator from X to Y , denoted by $F'(x)$ (or $DF(x)$) and called derivative of $F(x)$, such that

$$\forall h \in X \quad F(x + h) = F(x) + \langle F'(x), h \rangle + \epsilon(h) \|h\|_X \quad (7.24)$$

with $\lim_{h \rightarrow 0} \|\epsilon(h)\|_Y = 0$.

Remark: We are more interested in the case $Y = \mathbb{R}$. As it is known for functions, to be differentiable is more strong property than to be just derivable. The same thing is true for the operators.

More precisely, if $F(x) : X \rightarrow \mathbb{R}$ is differentiable (in the sense of Fréchet) at $x \in X$, then for all $h \in X$ there exists its directional derivative (usually also called the Gâteaux derivative). Indeed, from (7.24) with λh we find, since the application $F'(x)$ is linear and $\lambda \in \mathbb{R}$,

$$\frac{F(x + \lambda h) - F(x)}{\lambda} = \langle F'(x), h \rangle + \epsilon(\lambda h) \|h\|_X.$$

If $\lambda \rightarrow 0$, then $\epsilon(\lambda h) \rightarrow 0$ and hence $\frac{d}{d\lambda} F(x + \lambda h)|_{\lambda=0} = \langle F'(x), h \rangle$.

Attention: The notation $\langle F'(x), h \rangle$ is the same as for linear functionals: previously we denoted for a linear functional f by $\langle f, h \rangle$ the value of f on h , i.e. $f(h) = \langle f, h \rangle$. Here, for $Y = \mathbb{R}$, the derivative $F'(x)$ is a linear continuous functional from X to $\mathbb{R}$:

$$h \in X \mapsto \langle F'(x), h \rangle \in \mathbb{R}.$$

In other words, the directional derivative of $F(x)$ in the direction h $\langle F'(x), h \rangle$, it is the value of the linear functional $F'(x)$ on $h \in X$:

$$\langle F'(x), h \rangle = F'(x)(h).$$

In what follows, we need to understand in detail the following three examples.

Example 7.5.2. Let $F(x) : X \rightarrow \mathbb{R}$ be a linear continuous functional. Then

$$\forall x, h \in X \quad F(x + h) = F(x) + F(h).$$

Therefore, comparing to (7.24), we obtain that $\epsilon(h) = 0$ (the decomposition of $F(x + h)$ is exact) and $\langle F'(x), h \rangle = \langle F, h \rangle$, which is a linear continuous functional by the assumption for $F(x)$. It means that **derivative of a linear continuous functional is equal to itself**: $F'(x) = F$.

Example 7.5.3. Let $F(u) = \int_{\Omega} |\nabla u|^2 dx : H_0^1(\Omega) \rightarrow \mathbb{R}$. (Here we suppose that Ω is fixed and u is just an element of the space $H_0^1(\Omega)$).

Then

$$\forall u, h \in H_0^1(\Omega) \quad F(u + h) = F(u) + 2 \int_{\Omega} \nabla u \nabla h dx + F(h).$$

We can see that $h \mapsto 2 \int_{\Omega} \nabla u \nabla h dx$ is linear continuous functional on $H_0^1(\Omega)$ for a fixed $u \in H_0^1(\Omega)$. In addition, $F(h) = \int_{\Omega} |\nabla h|^2 dx = \|h\|_{H_0^1(\Omega)}^2$ and thus, taking $\epsilon(h) = \|h\|_{H_0^1(\Omega)}$ can be expressed as $\epsilon(h)\|h\|_{H_0^1(\Omega)}$ with $\lim_{h \rightarrow 0} \epsilon(h) = 0$. (Here $Y = \mathbb{R}$, then $\|\cdot\|_Y = |\cdot|$, but $\|h\|_{H_0^1(\Omega)}$ is all times positive. The convergence $h \rightarrow 0$ is understood as the convergence in $X = H_0^1(\Omega)$). Hence we conclude that $F(u)$ is Fréchet differentiable for all $u \in H_0^1(\Omega)$ and

$$\langle F'(u), h \rangle = 2 \int_{\Omega} \nabla u \nabla h dx.$$

Example 7.5.4. Let $F(u) = \int_{\Omega} |u|^2 dx : L^2(\Omega) \rightarrow \mathbb{R}$. (Here we suppose that Ω is fixed and u is just an element of the space $L^2(\Omega)$).

Then

$$\forall u, h \in L^2(\Omega) \quad F(u + h) = F(u) + 2 \int_{\Omega} u h dx + F(h).$$

We can see that $h \mapsto 2 \int_{\Omega} u h dx$ is linear continuous functional on $L^2(\Omega)$ for a fixed $u \in L^2(\Omega)$. In addition, $F(h) = \int_{\Omega} |h|^2 dx = \|h\|_{L^2(\Omega)}^2$ and thus, taking $\epsilon(h) = \|h\|_{L^2(\Omega)}$ can be expressed as

$\epsilon(h)\|h\|_{L^2(\Omega)}$ with $\lim_{h \rightarrow 0} \epsilon(h) = 0$. Hence we conclude that $F(u)$ is Fréchet differentiable for all $u \in L^2(\Omega)$ and

$$\langle F'(u), h \rangle = 2 \int_{\Omega} u h \, dx.$$

7.5.2 Application to parametric optimization

This Section summarizes the Lagrangian method to find the derivative of J^* over χ in the framework of the parametric optimization for the Helmholtz system (7.7) with $\eta = 0$ on Γ . For more details, see [3].

Lagrangian method: main idea

Let us have a variational formulation of a parametric problem on a fixed Ω , *i.e.* its solution u depends of the parameter χ . In the schematic way, for a Hilbert space H , $u \in H$, depending on χ , solve the (real) variational formulation

$$\forall v \in H \quad FV(\chi, u(\chi), v) = 0. \quad (7.25)$$

As in our case, we assume that the initial PDE problem is linear, and thus, the variational formulation is a sum of a bilinear form and a linear form (see (7.8)).

The text function v does not depend on χ . We minimize a functional $J(\chi, u(\chi)) = \int_{\Omega} |u(\chi)|^2 \, dx$ on an admissible class of χ (our $U_{ad}^*(\beta)$) and we need to derive it on χ . It means that J is a function of χ and we want to find its (Fréchet) derivative $J'(\chi)$, *i.e.* the linear continuous functional $\chi_0 \mapsto \langle J'(\chi), \chi_0 \rangle$ (see Definition 7.5.2).

The simplest way to find it is to introduce **the Lagrangian** of the considered optimization problem. We formally define

$$\mathcal{L}(\chi, w, q) = FV(\chi, w, q) + J(w), \quad \text{for all } w, q \in H, \quad (7.26)$$

i.e. the Lagrangian is the sum of the variational formulation and the minimizing functional for arbitrary arguments w and q in H which are

- independent (w does not depend on q and conversely),
- independent of χ .

Thus, **all variables of the Lagrangian $\mathcal{L}$ are independent.**

We notice that the directional partial derivative of $\mathcal{L}$ corresponding to q in the direction $\phi \in H$ is given by

$$\left\langle \frac{\partial \mathcal{L}}{\partial q}(\chi, w, q), \phi \right\rangle = FV(\chi, w, \phi), \quad (7.27)$$

as soon as $J(w)$ does not depend on q (and the variational formulation is linear on q).

Consequently,

$$\forall \phi \in H \quad \left\langle \frac{\partial \mathcal{L}}{\partial q}(\chi, w, q), \phi \right\rangle = 0 \quad (7.28)$$

is the variational formulation of the solving problem, giving the weak solution $u \in H$.

Now, if we chose $w = u(\chi)$ the solution of the variational problem, we have, by definition (7.26),

$$\mathcal{L}(\chi, u(\chi), q) = J(\chi), \quad \text{for all } q \in H. \quad (7.29)$$

Thus, using the rule of compound derivation, we find for all $q \in H$

$$\langle J'(\chi), \chi_0 \rangle = \left\langle \frac{\partial \mathcal{L}}{\partial \chi}(\chi, u(\chi), q), \chi_0 \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial w}(\chi, u(\chi), q), \langle u'(\chi), \chi_0 \rangle \right\rangle, \quad (7.30)$$

where we formally suppose that the mapping $\chi \rightarrow u(\chi)$ is Fréchet differentiable on $U_{ad}^*(\beta)$, i.e. its derivative $u'(\chi) : \chi_0 \in U_{ad}^*(\beta) \mapsto \langle u'(\chi), \chi_0 \rangle =: \psi \in H$ is linear continuous. With the notation $\langle u'(\chi), \chi_0 \rangle = \psi$, an element of H , Eq. (7.31) becomes

$$\langle J'(\chi), \chi_0 \rangle = \left\langle \frac{\partial \mathcal{L}}{\partial \chi}(\chi, u(\chi), q), \chi_0 \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial w}(\chi, u(\chi), q), \psi \right\rangle. \quad (7.31)$$

Therefore, the idea is to take $q \in H$ as the unique weak solution of the following variational problem (for a fixed χ and a fixed $u(\chi)$)

$$\forall \phi \in H \quad \left\langle \frac{\partial \mathcal{L}}{\partial w}(\chi, u(\chi), q), \phi \right\rangle = 0. \quad (7.32)$$

This variational problem is called **the adjoint problem** and its weak solution is denoted by $p(\chi)$. As (7.32) holds for all $\phi \in H$, it holds also for $\phi = \psi$. This simplify (7.31) and we conclude that

$$\langle J'(\chi), \chi_0 \rangle = \left\langle \frac{\partial \mathcal{L}}{\partial \chi}(\chi, u(\chi), p(\chi)), \chi_0 \right\rangle, \quad (7.33)$$

where $p(\chi)$ is the solution of the adjoint problem

$$\forall \phi \in H \quad \left\langle \frac{\partial \mathcal{L}}{\partial w}(\chi, u(\chi), p(\chi)), \phi \right\rangle = 0. \quad (7.34)$$

So, summarizing the derivation of J , we need

1. To write the real Lagrangian of the parametric optimization problem (and to verify that all variables are independent).
2. To find the partial derivative of the Lagrangian over the second (independent!) variable w and write $\frac{\partial \mathcal{L}}{\partial w}(\chi, u(\chi), q)$.
3. To find the adjoint system from (7.34).
4. Define $J'(\chi)$ using formula (7.33), where the Lagrangian is evaluated for $w = u(\chi)$ and $q = p(\chi)$.

And, if we want to be rigorous, we also need to prove that the mapping $\chi \rightarrow u(\chi)$ is Fréchet differentiable on $U_{ad}^*(\beta)$ (see [3, Lemme 5.15 p.89] and [14]). In the framework of our course this fact is admitted.

Application of the Lagrangian method to our problem

Attention: In this section, to avoid too long formulas, instead of $\text{Tr}u$, $\text{Tr}u_R$, $\text{Tr}u_I$, $\text{Tr}p$, $\text{Tr}p_R$, $\text{Tr}p_I$, ... on Γ , we simply write u , u_R , u_I , p , p_R , p_I respectively on Γ . But it needs to be understood in the sense of their traces.

We have a difficulty: our problem, in particular its solution u , is complex-valued, as it contains the complex coefficient α .

Thus to have a real Lagrangian we need to separate the real and imaginary parts in (7.7), knowing that

$$u = u_R + iu_I, \quad \alpha = \alpha_R + i\alpha_I, \quad 0 \leq \chi \leq 1, \quad k > 0, \quad f = f_R + if_I, \quad \eta = 0$$

and in addition

$$\alpha u = (u_R + iu_I)(\alpha_R + i\alpha_I) = \alpha_R u_R - \alpha_I u_I + i(\alpha_I u_R + \alpha_R u_I).$$

Thus the real part and the imaginary part of (7.7) are respectively

$$\begin{cases} (\Delta + k^2)u_R = f_R \text{ on } \Omega; \\ \frac{\partial u_R}{\partial n} = 0 \text{ on } \Gamma_{\text{Neu}}; \\ u_R = 0 \text{ on } \Gamma_{\text{Dir}}; \\ \frac{\partial u_R}{\partial n} + \chi(\alpha_R u_R - \alpha_I u_I) = 0 \text{ on } \Gamma; \end{cases}$$

$$\begin{cases} (\Delta + k^2)u_I = f_I \text{ on } \Omega; \\ \frac{\partial u_I}{\partial n} = 0 \text{ on } \Gamma_{\text{Neu}}; \\ u_I = 0 \text{ on } \Gamma_{\text{Dir}}; \\ \frac{\partial u_I}{\partial n} + \chi(\alpha_R u_I + \alpha_I u_R) = 0 \text{ on } \Gamma; \end{cases}$$

We find now the the variational formulations of these two problems:

$$\forall v_R \in V(\Omega) \quad - \int_{\Gamma} \chi(\alpha_R u_R - \alpha_I u_I) v_R d\mu - \int_{\Omega} \nabla u_R \nabla v_R dx + k^2 \int_{\Omega} u_R v_R dx = \int_{\Omega} f_R v_R dx, \quad (7.35)$$

and

$$\forall v_I \in V(\Omega) \quad - \int_{\Gamma} \chi(\alpha_I u_R + \alpha_R u_I) v_I d\mu - \int_{\Omega} \nabla u_I \nabla v_I dx + k^2 \int_{\Omega} u_I v_I dx = \int_{\Omega} f_I v_I dx. \quad (7.36)$$

Attention: The test functions v_R and v_I are independent of each other. Eqs. (7.35) and (7.36) are coupled: each of them contains both u_R and u_I .

As v_R and v_I are independent, we subtract (7.36) from (7.35) and obtain the following real variational formulation for all $(v_R, v_I) \in V(\Omega) \times V(\Omega)$

$$FV(\chi, u_R, u_I, v_R, v_I) := \int_{\Gamma} \chi [(\alpha_I u_R + \alpha_R u_I) v_I - (\alpha_R u_R - \alpha_I u_I) v_R] d\mu + \int_{\Omega} [\nabla u_I \nabla v_I - \nabla u_R \nabla v_R + k^2(u_R v_R - u_I v_I) + f_I v_I - f_R v_R] dx = 0.$$

Here u_R and u_I depend of χ .

We now define the Lagrangian of the optimization problem as the sum of the previously calculated variational formulation and the objective function (7.4)

$$\forall \chi \in L^{\infty}(\Gamma, \mu), \quad \forall w_R, w_I, q_R, q_I \in V(\Omega)$$

$$\mathcal{L}(\chi, w_R, w_I, q_R, q_I) = \int_{\Omega} (w_R^2 + w_I^2) dx + FV(\chi, w_R, w_I, q_R, q_I).$$

Here all arguments of the Lagrangian, χ, w_R, w_I, q_R, q_I are independent. In the explicit form the Lagrangian is

$$\begin{aligned} \mathcal{L}(\chi, w_R, w_I, q_R, q_I) = & \int_{\Omega} (w_R^2 + w_I^2) dx \\ & + \int_{\Gamma} \chi [(\alpha_I w_R + \alpha_R w_I) q_I - (\alpha_R w_R - \alpha_I w_I) q_R] d\mu + \int_{\Omega} [\nabla w_I \nabla q_I \\ & - \nabla w_R \nabla q_R + k^2(w_R q_R - w_I q_I) + f_I q_I - f_R q_R] dx. \end{aligned}$$

Let us find the adjoint problem. We start by calculating $\langle \frac{\partial \mathcal{L}}{\partial w_R}(\chi, u_R, u_I, q_R, q_I), \phi_R \rangle$ and $\langle \frac{\partial \mathcal{L}}{\partial w_I}(\chi, u_R, u_I, q_R, q_I), \phi_I \rangle$.

$$\begin{aligned} \langle \frac{\partial \mathcal{L}}{\partial w_R}(\chi, u_R, u_I, q_R, q_I), \phi_R \rangle = & \int_{\Omega} 2u_R \phi_R + \int_{\Gamma} \chi [\alpha_I q_I \phi_R - \alpha_R q_R \phi_R] d\mu \\ & + \int_{\Omega} [-\nabla q_R \nabla \phi_R + k^2 q_R \phi_R] dx, \end{aligned}$$

$$\begin{aligned} \langle \frac{\partial \mathcal{L}}{\partial w_I}(\chi, u_R, u_I, q_R, q_I), \phi_I \rangle = & \int_{\Omega} 2u_I \phi_I + \int_{\Gamma} \chi [\alpha_R q_I \phi_I + \alpha_I q_R \phi_I] d\mu \\ & + \int_{\Omega} [-\nabla q_I \nabla \phi_I - k^2 q_I \phi_I] dx. \end{aligned}$$

Remark: To obtain these formulas, we use the form of J , which is quadratic on u_R and u_I , respectively, and of the variational formulation FV , which is linear on u_R and u_I respectively. Thus, by Examples of Subsection 7.5.1, they are Fréchet differentiable. In addition, if, for example, we derive over u_R a term which does not depend on it, as $\int_{\Omega} u_I^2 dx$, then its derivative is equal to zero. When we derive the linear terms, for example, over u_R , then, roughly speaking, it is sufficient to replace in these terms u_R by ϕ_R , the direction of the derivative, to obtain the derivative of these linear terms.

We deduce the adjoint system from

$$\left\langle \frac{\partial \mathcal{L}}{\partial w_R}, \phi_R \right\rangle = 0 \quad \forall \phi_R \in V(\Omega) \quad \text{and} \quad \left\langle \frac{\partial \mathcal{L}}{\partial w_I}, \phi_I \right\rangle = 0 \quad \forall \phi_I \in V(\Omega).$$

Instead to keep the notation q_R and q_I denoting the independent variables of the Lagrangian, we denote the solution of the adjoint system by p_R and p_I . In the explicit form we have the following variational formulation for the adjoint problem

$$\begin{aligned} \forall \phi_R \in V(\Omega) \quad & \int_{\Omega} 2u_R \phi_R dx + \int_{\Gamma} \chi [\alpha_I p_I \phi_R - \alpha_R p_R \phi_R] d\mu + \int_{\Omega} [-\nabla p_R \nabla \phi_R + k^2 p_R \phi_R] dx = 0, \\ \forall \phi_I \in V(\Omega) \quad & \int_{\Omega} 2u_I \phi_I dx + \int_{\Gamma} \chi [\alpha_R p_I \phi_I + \alpha_I p_R \phi_I] d\mu + \int_{\Omega} [-\nabla p_I \nabla \phi_I - k^2 p_I \phi_I] dx = 0. \end{aligned}$$

Actually the variational formulations are sufficient, but it could be also helpful to see the corresponding strong problems, which could formally obtained with the help of integration by parts

$$\begin{cases} (\Delta + k^2)p_R = -2u_R \text{ on } \Omega; \\ \frac{\partial p_R}{\partial n} = 0 \text{ on } \Gamma_{Neu}; \\ p_R = 0 \text{ on } \Gamma_{Dir}; \\ \frac{\partial p_R}{\partial n} + \chi(\alpha_R p_R - \alpha_I p_I) = 0 \text{ on } \Gamma; \end{cases} \quad (7.37)$$

$$\begin{cases} (\Delta + k^2)p_I = 2u_I \text{ on } \Omega; \\ \frac{\partial p_I}{\partial n} = 0 \text{ on } \Gamma_{Neu}; \\ p_I = 0 \text{ on } \Gamma_{Dir}; \\ \frac{\partial p_I}{\partial n} + \chi(\alpha_I p_R + \alpha_R p_I) = 0 \text{ on } \Gamma; \end{cases} \quad (7.38)$$

From systems (7.37) and (7.38) we deduce the complex valued system for $p = p_R + ip_I$:

$$\begin{cases} \Delta p + k^2 p = -2\bar{u} \text{ on } \Omega; \\ \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma_{Neu}; \\ p = 0 \text{ on } \Gamma_{Dir}; \\ \frac{\partial p}{\partial n} + \chi \alpha p = 0 \text{ on } \Gamma; \end{cases} \quad (7.39)$$

We notice that p depends on χ .

Let us give some details how to pass from the variational formulation of the adjoint problem to its strong form (7.37) and (7.38). Let us take the first equation for the real part. First of all we separate the terms containing the integrals over Ω and the terms with the boundary integrals:

$$\begin{aligned} \forall \phi_R \in V(\Omega) \quad & \int_{\Omega} [2u_R \phi_R - \nabla p_R \nabla \phi_R + k^2 p_R \phi_R] dx \\ & + \int_{\Gamma} \chi [\alpha_I p_I \phi_R - \alpha_R p_R \phi_R] d\mu = 0. \end{aligned}$$

We need to obtain the following form:

$$\int_{\Omega} [\dots] \phi_R dx + \int_{\partial\Omega} [\dots] \phi_R d\mu = 0.$$

It would possible if we integrate by parts in the term $\int_{\Omega} \nabla p_R \nabla \phi_R dx$:

$$\int_{\Omega} \nabla p_R \nabla \phi_R dx = \int_{\partial\Omega} \frac{\partial p_R}{\partial \nu} \text{Tr} \phi_R d\mu - \int_{\Omega} \Delta p_R \phi_R dx.$$

This gives us (we omit Tr and suppose that $\frac{\partial p_R}{\partial \nu} \in L^2(\partial\Omega, \mu)$ and $\Delta p_R \in L^2(\Omega)$)

$$\begin{aligned} \forall \phi_R \in V(\Omega) \quad & \int_{\Omega} [2u_R + \Delta p_R + k^2 p_R] \phi_R dx \\ & - \int_{\partial\Omega} \frac{\partial p_R}{\partial \nu} \phi_R d\mu + \int_{\Gamma} \chi [\alpha_I p_I - \alpha_R p_R] \phi_R d\mu = 0. \end{aligned} \quad (7.40)$$

As $H_0^1(\Omega) \subset V(\Omega)$ and (7.40) holds for all ϕ_R from $V(\Omega)$, then it also holds for ϕ_R from $H_0^1(\Omega)$, i.e. with $\text{Tr} \phi_R = 0$ on $\partial\Omega$ μ -a.e.:

$$\forall \phi_R \in H_0^1(\Omega) \quad \int_{\Omega} [2u_R + \Delta p_R + k^2 p_R] \phi_R dx = 0.$$

This can be viewed as the orthogonality in $L^2(\Omega)$:

$$\forall \phi_R \in H_0^1(\Omega) \quad (2u_R + \Delta p_R + k^2 p_R, \phi_R)_{L^2(\Omega)} = 0.$$

With $G = 2u_R + \Delta p_R + k^2 p_R \in L^2(\Omega)$, we find by the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ and the continuity of the inner product that

$$\forall \phi \in L^2(\Omega) \quad (G, \phi)_{L^2(\Omega)} = 0.$$

Hence, G is orthogonal to all elements of $L^2(\Omega)$. Consequently, it is also orthogonal to itself. But $(G, G)_{L^2(\Omega)} = 0$ implies that $G = 0$ a.e. on Ω . This gives us the equation

$$\Delta p_R + k^2 p_R = -2u_R, \quad x \in \Omega.$$

Going back to (7.40), knowing that $G = 0$, we have

$$\forall \phi_R \in V(\Omega) \quad - \int_{\partial\Omega} \frac{\partial p_R}{\partial \nu} \text{Tr} \phi_R d\mu + \int_{\Gamma} \chi [\alpha_I p_I - \alpha_R p_R] \text{Tr} \phi_R d\mu = 0. \quad (7.41)$$

Knowing that $\text{Tr}(H^1(\Omega)) = B(\partial\Omega)$ dense in $L^2(\partial\Omega, \mu)$, we find

$$\forall \psi \in L^2(\partial\Omega, \mu) : \psi = 0 \text{ } \mu - \text{a.e. on } \Gamma_{Dir} \quad \left(-\frac{\partial p_R}{\partial \nu} + 1_{\Gamma} \chi [\alpha_I p_I - \alpha_R p_R], \psi \right)_{L^2(\partial\Omega, \mu)} = 0.$$

By the definition of $V(\Omega)$, where we have our initial variational formulation, $\text{Tr} p_R = 0$ on Γ_{Dir} μ -a.e.. As previously, for $S = -\frac{\partial p_R}{\partial \nu} + 1_{\Gamma} \chi [\alpha_I p_I - \alpha_R p_R] \in L^2(\partial\Omega, \mu)$ we obtain that $(S, S)_{L^2(\partial\Omega, \mu)} = 0$. Then we conclude that $\frac{\partial p_R}{\partial \nu} = 0$ on Γ_{Neu} but $\frac{\partial p_R}{\partial \nu} = \chi [\alpha_I p_I - \alpha_R p_R]$ on Γ μ -a.e.. So, we obtained (7.37). In the same way, we obtain also (7.38).

Finally, we calculate by (7.33) the derivative of J over χ , evaluated in the direction χ_0 :²

$$\begin{aligned} \langle J'(\chi), \chi_0 \rangle &= \left\langle \frac{\partial \mathcal{L}}{\partial \chi}(\chi, u_R(\chi), u_I(\chi), p_R(\chi), p_I(\chi)), \chi_0 \right\rangle \\ &= \int_{\Gamma} \chi_0 [\alpha_I u_R(\chi) + \alpha_R u_I(\chi)] p_I(\chi) - (\alpha_R u_R(\chi) - \alpha_I u_I(\chi)) p_R(\chi) d\mu \\ &= - \int_{\Gamma} \chi_0 \text{Re}(\alpha u(\chi) p(\chi)) d\mu. \end{aligned} \quad (7.42)$$

We notice that when we write here $\langle \frac{\partial \mathcal{L}}{\partial \chi}(\chi, u_R(\chi), u_I(\chi), p_R(\chi), p_I(\chi)), \chi_0 \rangle$ it means that we find the derivative $\langle \frac{\partial \mathcal{L}}{\partial \chi}(\chi, w_R, w_I, q_R, q_I), \chi_0 \rangle$ using the independence of all variables and then take $w_R = u_R(\chi)$, $w_I = u_I(\chi)$, $q_R = p_R(\chi)$, $q_I = p_I(\chi)$ (in the same way as we have done for $\langle \frac{\partial \mathcal{L}}{\partial w_R}(\chi, u_R, u_I, q_R, q_I), \phi_R \rangle$ and $\langle \frac{\partial \mathcal{L}}{\partial w_I}(\chi, u_R, u_I, q_R, q_I), \phi_I \rangle$).

²For the proof of the Fréchet differentiability of u on χ see [54, Lemma 3, p.10], it is not required and can be omitted in this course.

7.6 Numerical algorithm

The standard numerical algorithm of the parametric optimization [3, p.94] uses the gradient method for a fixed frequency ω_0 and for a fixed $\beta \in]0, 1[$ and the noise sources f, h, g .

The main idea is to update/change χ in the direction of negative values of $J'(\chi)$ since it means decreasing the energy as a function of χ .

The numerical method is the following (for more discussions and details, see [3, p.94] and for a modified version see [54]):

Let $\zeta_n > 0$ be the step of the gradient descent method at the iteration n .

- **Initialization:** we fix χ_0 in $\mathcal{U}_{ad}$;

- **Iterations:**

$$\chi_{n+1} = \mathcal{P}_{\mathcal{U}_{ad}^*(\beta)}(\chi_n - \zeta_n J'(\chi_n)),$$

with $\mathcal{P}_{\mathcal{U}_{ad}^*(\beta)}$ a projection on $\mathcal{U}_{ad}^*(\beta)$ to determine (*i.e.* $\mathcal{P}_{\mathcal{U}_{ad}^*(\beta)} : (\Gamma \rightarrow \mathbb{R}) \rightarrow (\Gamma \rightarrow [0, 1])$) such as for all $\chi : \Gamma \rightarrow \mathbb{R}$ it holds $\frac{1}{\mu(\Gamma)} \int_{\Gamma} \mathcal{P}_{\mathcal{U}_{ad}^*(\beta)}(\chi) d\mu = \beta$). The following projector can be used according to [3]:

$$\mathcal{P}_{\mathcal{U}_{ad}^*(\beta)}(\chi) = \max(0, \min(\chi + \ell, 1)),$$

where ℓ is a constant to be determined at each iteration to ensure that the condition on the volume ($\frac{1}{\mu(\Gamma)} \int_{\Gamma} \chi d\mu = \beta$) is verified.

- **End:** If $|\chi_{n+1} - \chi_n| < \delta$ with an error δ chosen arbitrary small.

More precisely, to calculate χ_{n+1} we use the derivative of J in χ_n according to the found formula:

$$\begin{cases} \chi_0 \in \mathcal{U}_{ad} \\ \chi_{n+1}(x) = \max(0, \min(\chi_n(x) + \zeta_n \operatorname{Re}(\alpha p(\chi_n) u(\chi_n))|_{\Gamma}(x) + \ell_n, 1)) \in \mathcal{U}_{ad}^*(\beta) \end{cases}$$

The determination of ℓ_n can be performed by the dichotomy.

The steps of the gradient descent method $(\zeta_n)_{n \in \mathbb{N}}$ can be calculated in different ways, this calculation is proposed in [3]:

- **Initialization:** $\zeta_0 = c > 0$;

- **Iterations:** If $J(\chi_n) > J(\chi_{n+1}) : \zeta_{n+1} = \zeta_n + 0.001$;

$$j = 0;$$

$$\text{While } J(\chi_n) < J(\chi_{n+1}) : \zeta_{n+1} = \zeta_n / 2^{j+1}$$

$$j = j + 1.$$

If the algorithm stopped on the iteration N , we obtain the optimal distribution $\chi_N =: \chi^{opt} \in \mathcal{U}_{ad}^*(\beta)$. Then we need to project it on the set of characteristic functions, *i.e.* on $\{0, 1\}$ to obtain the final distribution $\chi_{projected}^{opt} = \mathcal{P}_{\mathcal{U}_{ad}(\beta)}(\chi^{opt})$ belonging this time to $\mathcal{U}_{ad}(\beta)$, as initially desired. Here, the projection operator $\mathcal{P}_{\mathcal{U}_{ad}(\beta)}$ is to define. For example, it is sufficient to sort all values of χ and set the largest ones to 1 and the rest to 0 to reach the desired volume $\int_{\Gamma} \chi d\mu$.

Remark: The choice of the projection $\mathcal{P} : (\Gamma \rightarrow \mathbb{R}) \rightarrow (\Gamma \rightarrow [0, 1])$ on $U_{ad}^*(\beta)$ with the condition

$$\frac{1}{\mu(\Gamma)} \int_{\Gamma} \mathcal{P}(\chi) d\mu = \beta, \quad \beta \in]0, 1[\quad (7.43)$$

is obviously not unique. Indeed, it can be viewed as the map

$$\mathcal{P} : \chi \mapsto \Phi \circ (\chi + \ell_{\chi}) \quad (7.44)$$

with ℓ_{χ} ensuring (7.43) with different choice of the function Φ :

a) Rectified linear projection:

$$\Phi(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \in [0, 1] \\ 1 & \text{if } x > 1 \end{cases}$$

b) Sigmoid projection:

$$\Phi(x) = \frac{1}{1 + \exp(-a(x - \frac{1}{2}))}, \quad a \in \mathbb{R}_+^*$$

c) Flattened linear projection:

$$\Phi(x) = \begin{cases} \delta \exp(\frac{x}{\delta} - 1) & \text{if } x < \delta \\ x & \text{if } x \in [\delta, 1 - \delta] \\ 1 - \delta \exp(\frac{1-x}{\delta} - 1) & \text{if } x > 1 - \delta \end{cases}, \quad \delta \in \mathbb{R}_+^*.$$

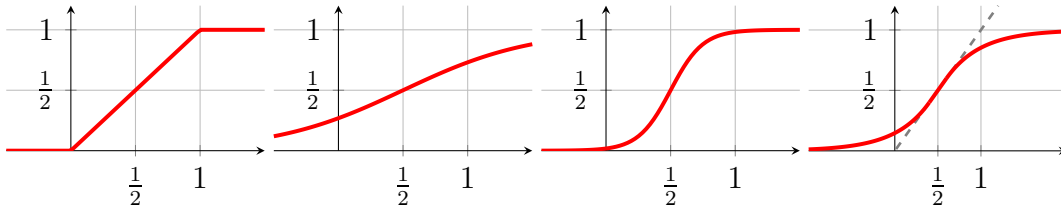


Figure 7.3 – Functions Φ for different projections (from left to right): Rectified linear, Sigmoid with $a = 2$; Sigmoid with $a = 8$; Flattened linear with $\delta = 0.4$.

7.7 Several numerical results

Let us give several numerical results obtained by M. Menoux [54] using the sigmoid projection with $a = 8$. To calculate the optimal absorbent distribution χ , we apply the Barzilai-Borwein gradient method with modified step size, inspired by Newton's method [11, 19]. For $\beta = 0.5$, we project the calculated $\chi^{opt} \in U_{ad}^*(0.5)$ on $U_{ad}(0.5)$ to obtain a characteristic function by replacing Φ in (7.44) by Φ_f :

$$\Phi_f(x) = \begin{cases} 0 & \text{if } x \leq 0.5, \\ 1 & \text{if } x > 0.5. \end{cases}$$

For instance, we aim to minimize the acoustical energy of an office (or a music hall) with a

perforated wall; see Fig. 7.4 for the model description. We suppose that the porous material used in the wall is ISOREL (see Chapter 3). We solve the problem using the centered finite differences

$$\begin{array}{c}
 \frac{\partial u}{\partial n}|_{\Gamma_{Neu}} = 0 \\
 \\
 \begin{array}{ccc}
 u|_{\Gamma_{Dir}} = g(y) & \Delta u + k^2 u = f \text{ on } \Omega & \frac{\partial u}{\partial n} + \alpha(\omega)\chi u|_{\Gamma} = 0
 \end{array} \\
 \\
 \frac{\partial u}{\partial n}|_{\Gamma_{Neu}} = 0
 \end{array}$$

Figure 7.4 – Optimization problem for a music hall.

with a regular mesh on a square domain $\Omega =]0, 1[^2$ (the unknowns are in the center of cells, it corresponds to the 2D finite volumes) and denote by h the size of the spatial discretization (for its detailed description, see Chapter B).

We just briefly summarize it here. In this study, $\Omega =]0, 1[\times]0, 1[$ is the square domain represented in figure 7.5 along with the other boundaries.

$$(P) : \begin{cases} (\Delta + k^2)u &= 0 \text{ on } \Omega, \\ u &= g \text{ on } \Gamma_D, \\ \partial_n u &= 0 \text{ on } \Gamma_N, \\ \partial_n u + \alpha \chi u &= 0 \text{ on } \Gamma. \end{cases}$$

Figure 7.6 – Studied model

Figure 7.5 – Geometry of the problem

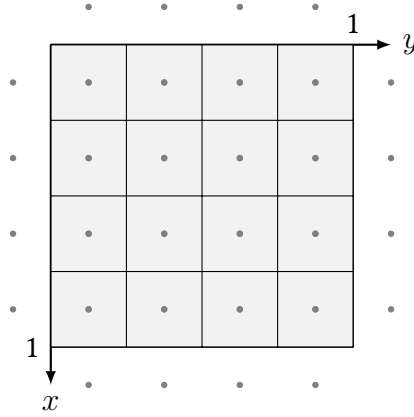
To approximate the solution of the Helmholtz problem, we use the cell-centered finite difference method (see Figure 7.7).

We divide the interval $[0, 1]$ into M equal parts, so there are M^2 small squares in Ω . Let $h = \frac{1}{M}$ and

$$\forall i, j \in \llbracket -1, M \rrbracket, \quad x_i = (i + \frac{1}{2})h, \quad y_j = (j + \frac{1}{2})h,$$

where (x_i, y_j) is the center with respect to each cell (i, j) of the mesh. Let $u_{i,j} = u(x_i, y_j)$. The points outside Ω are used to handle the boundary conditions.

We discretize the Helmholtz equation using a three points scheme:

Figure 7.7 – Mesh for $M = 4$

$$\forall i, j \in \llbracket 0, M-1 \rrbracket, \quad \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + k^2 u_{i,j} = 0. \quad (7.45)$$

All the boundary conditions can be expressed using three parameters: $a\partial_n u + bu = r$.

For the left boundary :

$$\forall i \in \llbracket 0, M-1 \rrbracket, a_{i,0} \frac{u_{i,0} - u_{i,-1}}{h} + b_{i,0} \frac{u_{i,-1} + u_{i,0}}{2} = r_{i,0}. \quad (7.46)$$

We discretize the other boundary conditions similarly (see Chapter B).

Finally, we solve the linear system to obtain an approximation of $u : (u_{i,j})_{(i,j) \in \llbracket 0, M-1 \rrbracket^2}$.

Attention: The mesh must be sufficiently small to correctly discretize a wave of a fixed wavelength λ and the geometrical inclusions of the porous material on Γ . Typically we need to have not less than 10 mesh points in one wavelength and in porous inclusions, *i.e.* to ensure $h \leq \frac{\min(\lambda, \min \text{len}(\chi))}{10}$ with $\min \text{len}$ the minimum of connected parts length of $\chi^{-1}(\{0\})$ and $\chi^{-1}(\{1\})$.

We set the following initial parameters

Parameter	Value	Description
Ω	$]0, 1[\times]0, 1[$	domain
c	340 m s^{-1}	wave speed
h	$\min(\lambda, \min \text{len}(\chi))/80$	spacestep
g	$(x, y = 0) \mapsto \exp \frac{(x-1/2)^2}{\sqrt{2\pi}0.5^2}$	Dirichlet boundary function
I	$(20 \text{ Hz}, 1000 \text{ Hz})$	chosen audible frequencies
β	50 %	fixed absorbent percentage
χ_0	see fig. 7.8	initial absorbent distribution

Because of a limited computational power, we stay below 1000 Hz and set $\text{minlen}(\chi)$ to 0.1 when $\chi \in [0, 1]$.

Attention: From now on, we denote by f the frequency, related with ω by the formula $\omega = 2\pi f$, and not a source term in the Helmholtz equation, which in this numerical test was taken equal to 0.

The parameter β is fixed to 0.5 (there is 50% of absorbing material on Γ). We also fix our initial distribution χ_0 , corresponding to the defined β , presented in Fig. 7.8 (on the left). The corresponding solution (its real and imaginary parts) for $f = 500$ Hz is given on the right-hand part of Fig. 7.8

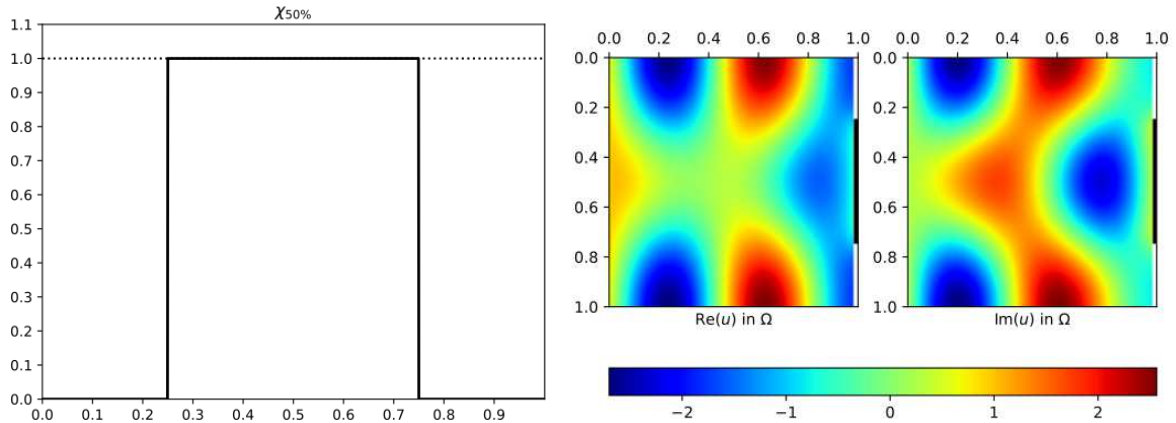


Figure 7.8 – $\chi(y)$ for the initial absorbent distribution on the left and the corresponding real and imaginary parts of the solution of the Helmholtz problem found for the frequency $f = 500$ Hz on the right. A black line gives the presence of the porous medium on Γ .

Fig. 7.9 gives the energy $J(\chi_0)$ for the chosen χ_0 (see Fig. 7.8 on the left) and the energy for the fully absorbent Γ ($\chi(x) = 1$ for all $x \in \Gamma$). The main goal is to find a distribution $\chi \in U_{ad}(0.5)$ such that the energy $J(\chi)$ on the interval of chosen frequencies would be not worse or even smaller than the energy corresponding to the fully absorbent Γ .

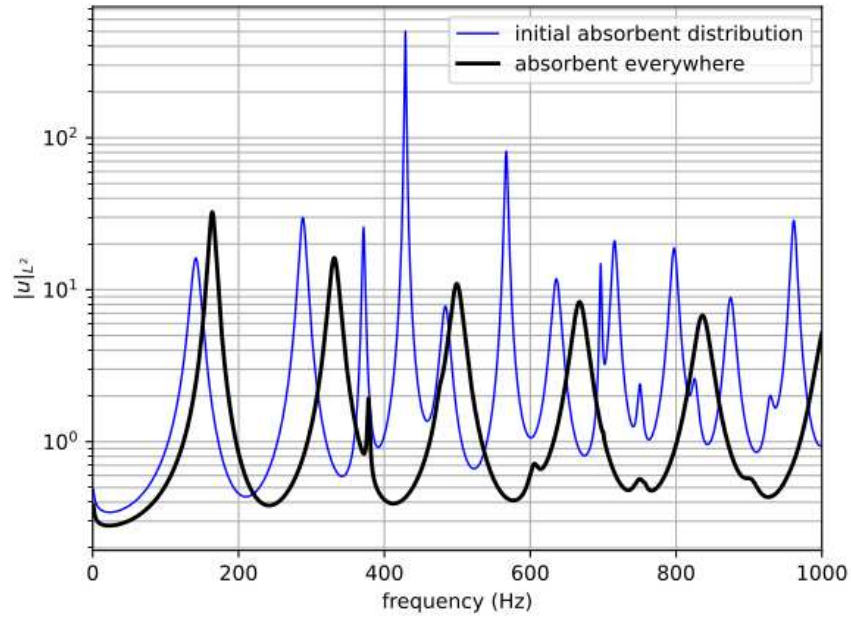


Figure 7.9 – energy on the frequency range for the initial absorbent distribution and the one with absorbent everywhere.

First, we consider the optimization for only one fixed frequency and then consider a frequency discretization of the interval $I = [20 \text{ Hz}, 1000 \text{ Hz}]$.

7.7.1 Optimization on a single frequency

We chose three typical frequencies: $f = 100 \text{ Hz}$ in the low frequencies, $f = 500 \text{ Hz}$ in the mean frequencies and $f = 1000 \text{ Hz}$ as a high frequency. Each time, the optimization procedure is started with χ_0 and $\beta = 0.5$, as explained previously.

Example $f = 100$ Hz in the low frequencies

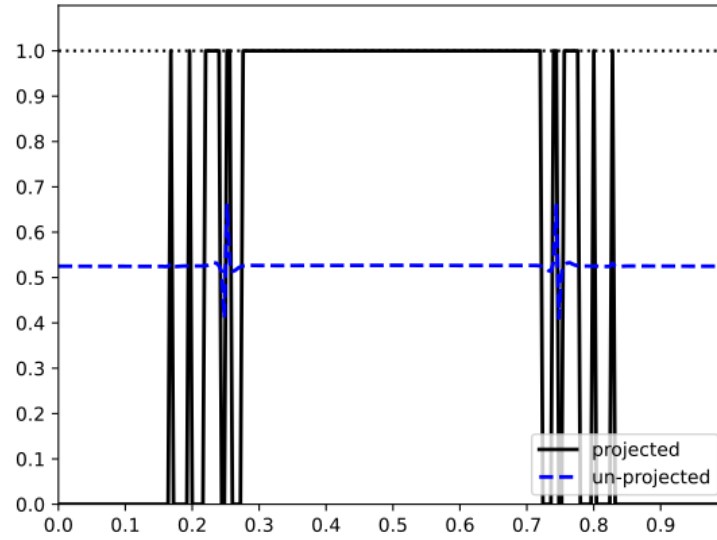


Figure 7.10 – $\chi^{opt} \in U_{ad}^*(0.5)$ (blue dotted line) and $\chi_{projected}^{opt} \in U_{ad}(0.5)$ (black solid line) for the frequency $f = 100$ Hz.

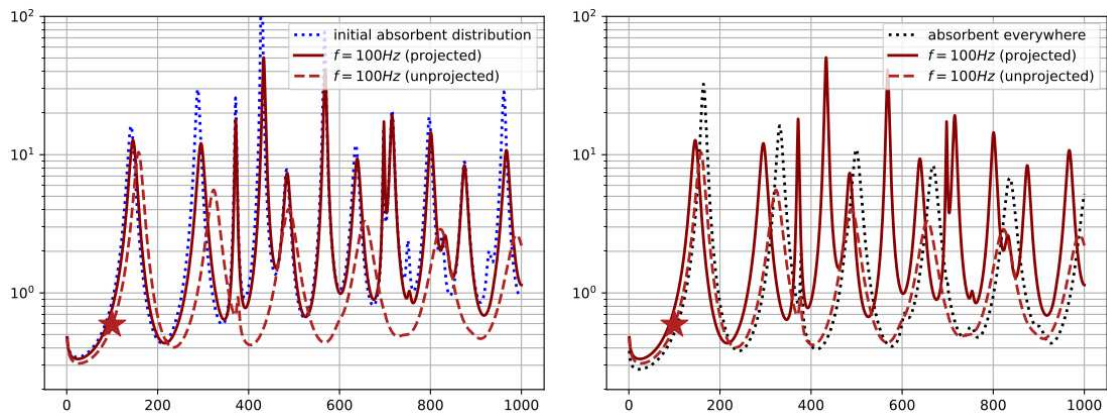


Figure 7.11 – Comparison between the energy for the optimised χ for the frequency $f = 100$ Hz and the initial χ_0 on the left and the fully absorbing case $\chi_{100\%}$ on the right.

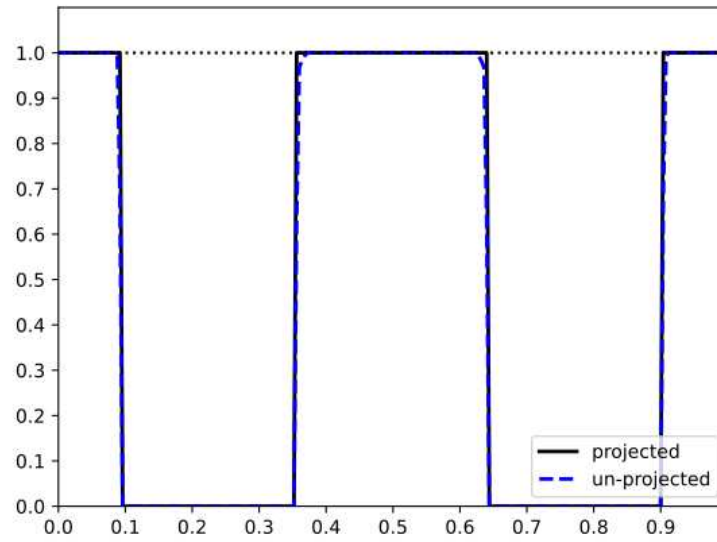
Example $f = 500$ Hz in the mean frequencies

Figure 7.12 – Optimal distribution $\chi^{opt} \in U_{ad}^*(0.5)$ and its projection on $U_{ad}(0.5)$ for the fixed frequency $f = 500$ Hz.

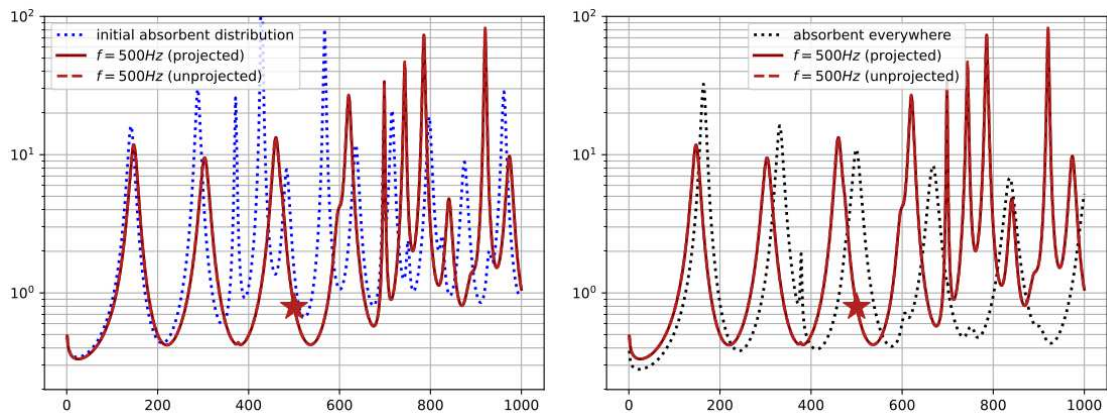


Figure 7.13 – Comparison between the energy for the optimised χ for the frequency $f = 500$ Hz and the initial χ_0 on the left and the fully absorbing case $\chi_{100\%}$ on the right.

Example $f = 1000$ Hz in the high frequencies

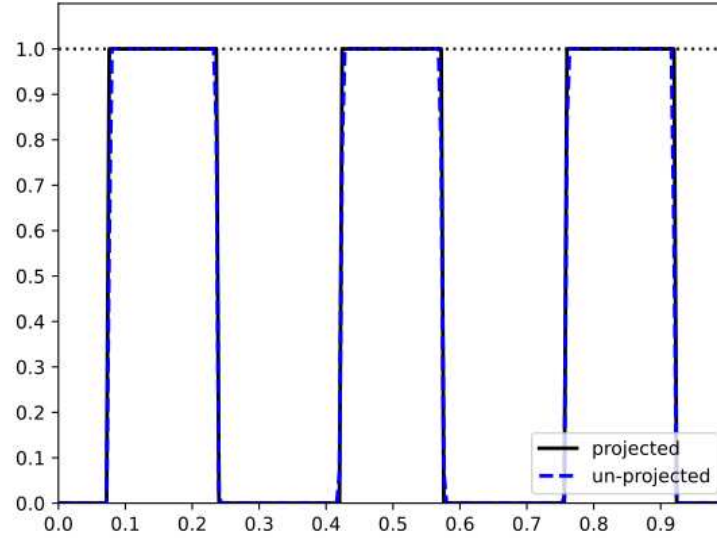


Figure 7.14 – Optimal distribution $\chi^{opt} \in U_{ad}^*(0.5)$ and its projection on $U_{ad}(0.5)$ for the fixed frequency $f = 1000$ Hz.

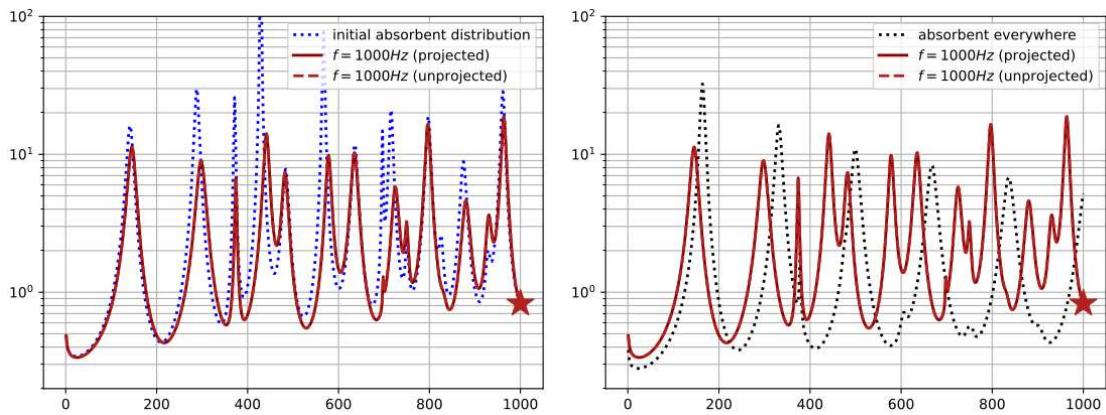


Figure 7.15 – Comparison between the energy for the optimised χ for the frequency $f = 1000$ Hz and the initial χ_0 on the left and the fully absorbing case $\chi_{100\%}$ on the right. In the mean, the energy found for the optimized χ for the frequency $f = 1000$ Hz with 50% of porous material is not too larger than the energy of the fully absorbing case.

Comparison of the different energies on the range of frequencies

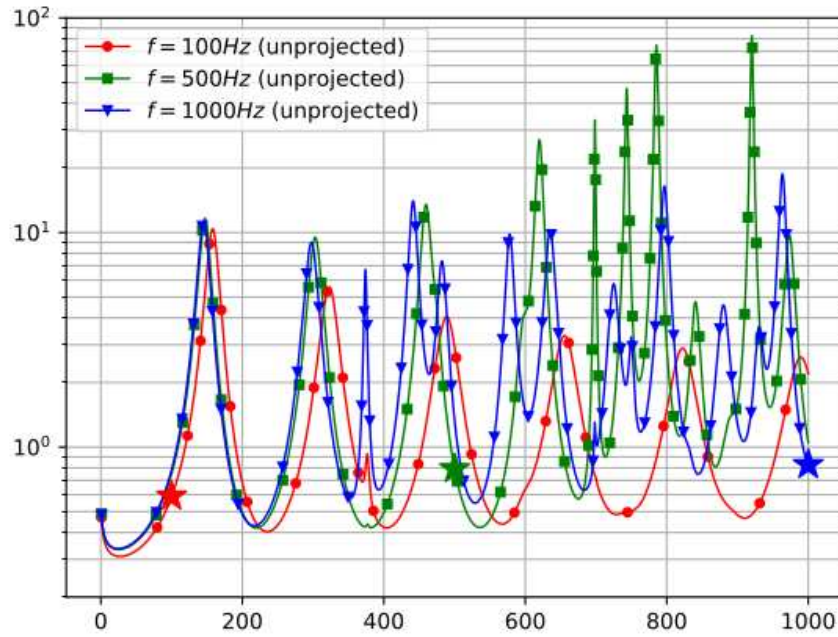


Figure 7.16 – Energies on the frequency range for three found absorbing distributions of χ^{opt} at 100 Hz (red line), 500 Hz (green line) and 1000 Hz (blue line). A star indicates the frequency of the optimization.

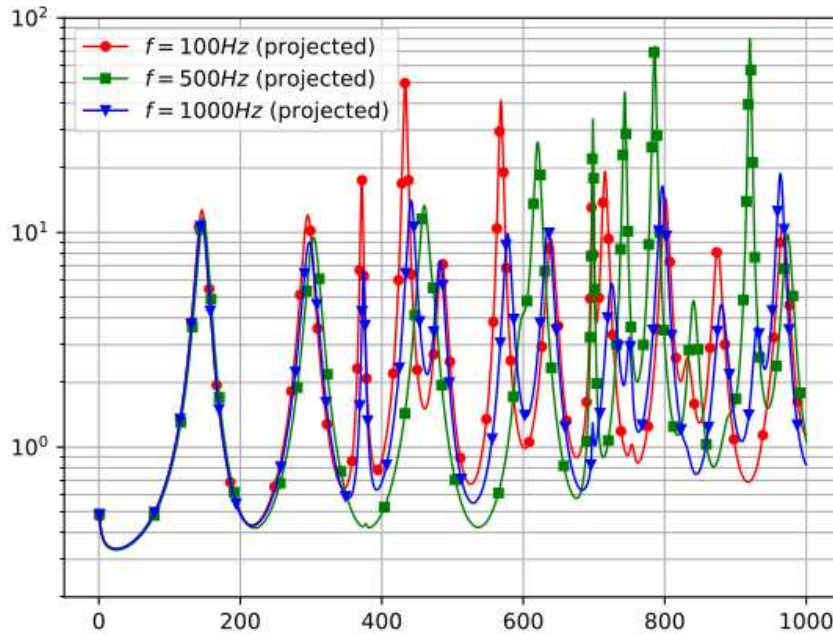


Figure 7.17 – Energies on the frequency range for three found absorbing distributions of $\chi_{projected}^{opt}$ at 100 Hz (red line), 500 Hz (green line) and 1000 Hz (blue line).

For $f \in \{500 \text{ Hz}, 1000 \text{ Hz}\}$, the optimization procedure tends to give optimal distributions χ^{opt} close to characteristic functions. Moreover, see, for instance, Fig. 7.12 for $f = 500 \text{ Hz}$, the width

of the peaks of the optimal χ is closed to $\frac{\lambda}{2} = \frac{c}{2f}$ (as it was obtained in the framework of the geometrical shape optimization). For $f = 100$ Hz, the expected peak length is higher than 1, which is impossible to reach with a volume percentage of $\beta = 50\%$.

We notice that the optimization on one frequency results in the displacement of the peaks without significantly reducing the total integrated energy. Thus, there is a need to optimize on several frequencies at once.

7.7.2 Optimization on several frequencies

From now on, we use 400 uniformly spaced frequencies on $I = [20Hz, 1000Hz]$. Then, we optimize the acoustical energy integrated over I . More precisely, we minimize the “total” energy on $U_{ad}(0.5)$ (see [54] for more details):

$$\hat{J}(\chi) := \int_I J(f, \chi) df, \quad \min_{\chi \in U_{ad}(0.5)} \hat{J}(\chi). \quad (7.47)$$

The integral is approximated by the (composite) trapezoidal rule element-wise, *i.e.*

$$\int_{f_j}^{f_{j+1}} J(\chi, f) df \approx (f_{j+1} - f_j) \frac{J(\chi, f_j) + J(\chi, f_{j+1})}{2}. \quad (7.48)$$

To see if the gradient descent gives good performances to reach an optimal distribution, we implemented a genetic algorithm (not detailed here, based on [2]) and compared its best result to the one we have (figure 7.19). We found approximately the same total energy.

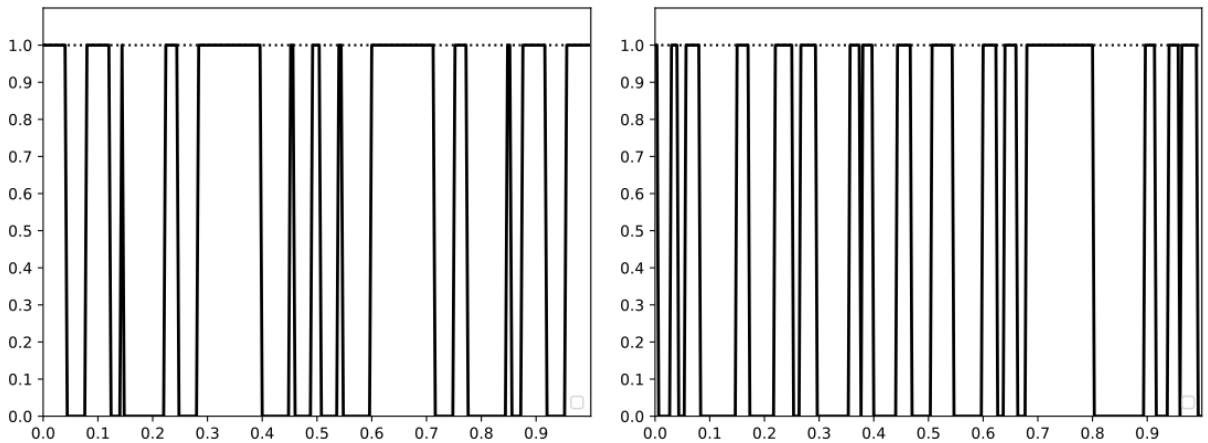


Figure 7.18 – Absorbent distribution χ found by gradient descent method on the left and genetic algorithm on the right for the frequency range optimization with 50% of porous material.

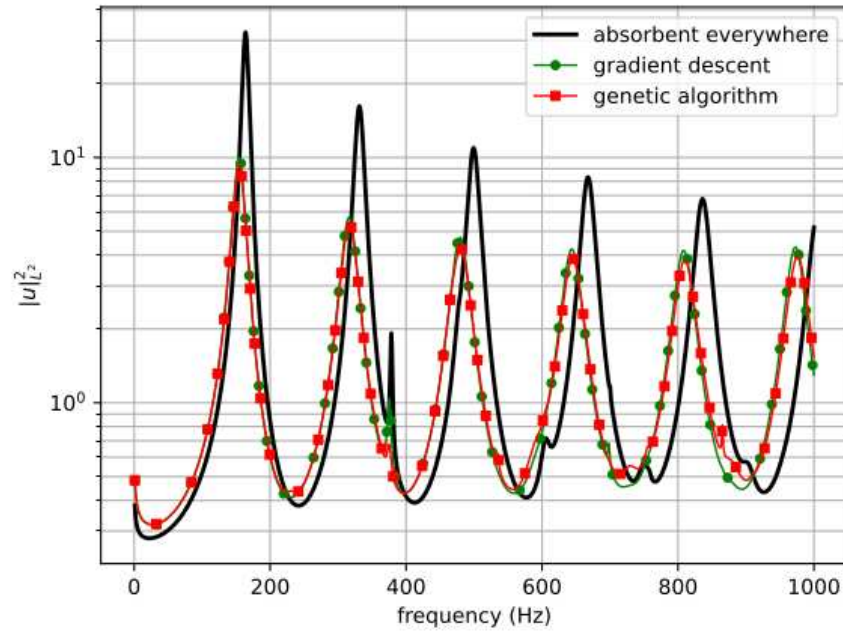


Figure 7.19 – Comparison between the best absorber distribution found using the gradient descent and the best distribution found using the genetic algorithm. We notice that they are almost the same and both better than the 100% absorbing boundary.

We give now the values of integrals over I for all energies presented on Fig. 7.19:

Distribution	Total energy
Absorbent everywhere	2093
Best using gradient descent	1475
Best using genetic algorithm	1440

Fig. 7.19 shows that with 50% of absorbing material, it is possible to create a distribution of porous material that absorbs more efficiently the acoustical energy than with 100% of absorbing material.

Appendix A

Remarks on the well-posedness of the Helmholtz problem

Let us consider the Helmholtz equation

$$\Delta u + \omega^2 u = f$$

with some boundary conditions on a bounded domain Ω . Its well-posedness is directly related to the question: is 0 an eigenvalue of the operator $\Delta + \omega^2$, or equivalently, is ω^2 an eigenvalue of Δ ?

In the case of considered complex-valued Robin boundary condition, the answer is negative (all eigenvalues are non-zero imaginary parts), and hence the problem is well-posed.

To prove its well-posedness, it is not possible to use the Lax-Milgram Theorem. The variational formulation of the Helmholtz problem gives a form $a(u, v) : H \times H \rightarrow \mathbb{R}$ which is not coercive (the problem comes from the term $\omega^2 u$). It is sufficient to consider the simplest case of the homogeneous Dirichlet boundary conditions: for all $v \in H_0^1(\Omega)$

$$(u, v)_{H_0^1(\Omega)} - \omega^2 (u, v)_{L^2(\Omega)} = -(f, v)_{L^2(\Omega)}.$$

Here, we have the form $a(u, v) = (u, v)_{H_0^1(\Omega)} - \omega^2 (u, v)_{L^2(\Omega)}$. Since ω is an arbitrary real number, the sign $-$ between two terms does not ensure its coercivity and positivity for $u = v$, since ω is an arbitrary real number. However, this kind of form can be rewritten in the sense of the operator equation (see TD 4):

$$\forall v \in H_0^1(\Omega) \quad ((Id - \omega^2 T)u, v)_{H_0^1(\Omega)} = -(Af, v)_{H_0^1(\Omega)},$$

involving a compact linear operator $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ (and a bounded linear operator $A : L^2(\Omega) \rightarrow H_0^1(\Omega)$).

It means that the well-posedness of the Helmholtz problem is equivalent to the well-posedness of the operator equation on a Hilbert space H (for the Dirichlet problem here $H = H_0^1(\Omega)$, for the mixte problem $H = V(\Omega), \dots$):

$$(Id - T)u = f \quad (f := -Af \text{ to simplify the notations}) \tag{A.1}$$

with a linear compact operator $T : H \rightarrow H$.

Equation (A.1) is called the Fredholm equation of the second kind¹. To solve it, there are three Fredholm theorems (this time, the Lax-Milgram is useless).

¹The Fredholm equation of the first kind has the form $Tu = f$.

A.1 Fredholm theory and theorems

Three Fredholm theorems are studying the properties of the operator $Id - T$ and the adjoint operator $Id - T^*$. We formulate them in the Hilbert space H framework, but they are also valid on a Banach space. The proof of these theorems is classical but involves the abstract theory of the functional analysis not treated in this course. Some elements of the proof are presented on TD 4.

In particular, we take attention to these two useful facts (we use them on TD 4):

1. If $T : H \rightarrow H$ is a linear compact operator, then its adjoint operator $T^* : H \rightarrow H$ is also compact.
2. If $T : H \rightarrow H$ is a linear compact operator, then $\text{Im}T = T(H)$ is a close set in H (see Proposition 6.2.1):

$$\overline{\text{Im}T} = \text{Im}T.$$

This is the first Fredholm theorem:

Theorem A.1.1. *Let T be a linear compact operator in a Hilbert space H . The following assertions are equivalent:*

1. *for all $f \in H$ equation $(Id - T)u = f$ has a solution in H (i.e. $\text{Im}(Id - T) = H$);*
2. *homogenous equation $(Id - T)z = 0$ has only the trivial solution $z = 0$ (i.e. $\text{Ker}(Id - T) = \{0\}$);*
3. *for all $g \in H$ the adjoint equation $(Id - T^*)v = g$ has a solution in H (i.e. $\text{Im}(Id - T^*) = H$);*
4. *homogenous equation $(Id - T^*)\phi = 0$ has only the trivial solution $\phi = 0$ (i.e. $\text{Ker}(Id - T^*) = \{0\}$).*

If one of the assertions 1)-4) holds, then there exist the inverse operators $(Id - T)^{-1}$ and $(Id - T^)^{-1}$, which are linear continuous.*

It also holds the second Fredholm theorem:

Theorem A.1.2. *Let H be a Hilbert space and $T : H \rightarrow H$ be a compact linear operator. Then equations $(Id - T)z = 0$ and $(Id - T^*)\phi = 0$ have the same finite number of linearly independent solutions.*

Finally, we give the third Fredholm theorem:

Theorem A.1.3. *Let H be a Hilbert space and $T : H \rightarrow H$ be a linear compact operator. Equation $(Id - T)u = f$ (with a fixed $f \in H$) has at least one solution in H iff for all solution $\phi \in H$ of $(Id - T^*)\phi = 0$ we have*

$$(\phi, f)_H = 0.$$

Therefore, to apply the first Fredholm theorem to the Helmholtz problem, we must ensure that the homogeneous problem ($f = 0$ and with homogeneous boundary conditions) has the unique solution $u = 0$.

For the homogeneous Dirichlet boundary conditions, it is possible to use the fact that $-\Delta$ with Dirichlet boundary conditions is a positive operator (also symmetric and auto-adjoint) with strictly positive eigenvalues. Since 0 is not its eigenvalue, thus the homogeneous problem has the unique trivial solution.

It is not the case of the Neumann Laplacian, which has an eigenvalue equal to 0. Obviously, any constant solves $\Delta u = 0$ with $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. In this case the third Fredholm Theorem is useful. Actually, if we precise $\int_{\Omega} u dx = 0$ (i.e. $(u, 1)_H = (u, 1)_{L^2(\Omega)} = 0$), we work with solutions of the mean value 0 on Ω , which allows only one possibility of the constant solution $u = 0$. In other words, the condition $\int_{\Omega} u dx = 0$ ensures that $\text{Ker}(Id - T) = \{0\}$ and gives the class of uniqueness: $u \in H$ of the mean value zero.

For the mixte boundary value problem with the complex Robin boundary conditions, the unicity of the homogeneous problem comes from the uniqueness of the solution to the Cauchy problem for $\Delta + \omega^2 Id$ in the connected domain Ω with Cauchy data on Γ (see for example [20, Theorems 1.1 and 1.2] and [53]). Actually, choosing the test function equal to a solution u and separating real and imaginary parts in the variational formulation of the homogeneous Helmholtz problem, we first obtain that $\text{Tr } u = 0$ on Γ (since $|\text{Im } \alpha| > 0$). By the Robin boundary condition on Γ , we then obtain that $\frac{\partial u}{\partial n} = 0$ on Γ . Thus on Γ we have simultanelly $\text{Tr } u = 0$ and $\frac{\partial u}{\partial n} = 0$ for u solving the second order equation $(\Delta + \omega^2)u = 0$ in Ω (with, as usual $\text{Tr } u = 0$ on Γ_{Dir} and $\frac{\partial u}{\partial n} = 0$ on Γ_{Neu}). It gives us the so-called the Cauchy problem for the operator $\Delta + \omega^2 Id$. We do not detail the proof of the unicity. Finally, we apply the first Fredholm theorem and conclude the well-posedness (see TD4).

Appendix B

Centered Finite Differences for the Helmholtz problem

Assume that the domain D be a rectangle, or simply a square. For $(x, y) \in D$, put

$$R(x, y) = \begin{cases} -1 & \text{if } (x, y) \in \Omega, \\ 1 & \text{if } (x, y) \in D \setminus \Omega. \end{cases}$$

This matrix allows to identify if (x, y) is inside or outside of Ω .

Assume that the square is $(0, 1) \times (0, 1)$, then we divide the interval $(0, 1)$ into N equal parts, so there are $N \times N$ small squares in D . It is called a mesh and each small square in mesh is called a cell.

Let $h = \frac{1}{N}$, for all $i \in (1, N)$ and all $j \in (1, N)$, put $x_i = (i - \frac{1}{2})h$, $y_j = (j - \frac{1}{2})h$, that means (x_i, y_j) is the center with respect to each cell which we will call the cell (i, j) in mesh. Let $u(i, j)$ be an approximate value of $u(x_i, y_j)$, and $f_{i,j} = f(x_i, y_j)$. Moreover, we build a matrix s which defines the value on the Robin boundary

$$\begin{aligned} s_{(2i+1, 2j)} &= s(x_i + h/2, y_j), \\ s_{(2i-1, 2j)} &= s(x_i - h/2, y_j), \\ s_{(2i, 2j+1)} &= s(x_i, y_j + h/2), \\ s_{(2i, 2j-1)} &= s(x_i, y_j - h/2). \end{aligned}$$

The size of matrix s is $(2N + 1) \times (2N + 1)$. We assume that the domain Ω is the interior of the square D . Now we discretize the Helmholtz problem as following.

- For $i \in (2, N - 1)$ and $j \in (2, N - 1)$, using the Helmholtz equation

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \omega^2 u_{i,j} = f_{i,j} \quad .$$

- For $i = 1$ and $j \in (2, N - 1)$, using the Helmholtz equation and the Dirichlet boundary condition

$$\frac{u_{i+1,j} - 3u_{i,j} + 2g(j)}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \omega^2 u_{i,j} = f_{i,j} \quad .$$

- For $i = 1$ and $j = 1$ or $j = N$, using the Helmholtz equation and the Dirichlet and Neumann boundary conditions,

- Case $j = 1$,

$$\frac{u_{i+1,j} - 3u_{i,j} + 2g(j)}{h^2} + \frac{u_{i,j+1} - u_{i,j}}{h^2} + w^2 u_{i,j} = f_{i,j} \quad .$$

- Case $j = N$,

$$\frac{u_{i+1,j} - 3u_{i,j} + 2g(j)}{h^2} + \frac{u_{i,j-1} - u_{i,j}}{h^2} + w^2 u_{i,j} = f_{i,j} \quad .$$

- For $i = N$ and $j = 1$ or $j = N$, using the Helmholtz equation and the Dirichlet, Neumann and Robin boundary conditions

- Case $j = 1$,

$$\frac{\frac{-\alpha h}{1 + \alpha h/2} u_{i,j} - u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - u_{i,j}}{h^2} + \omega^2 u_{i,j} = f_{i,j} + s_{(2i+1,2j)} \quad .$$

- Case $j = N$,

$$\frac{\frac{-\alpha h}{1 + \alpha h/2} u_{i,j} - u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j-1} - u_{i,j}}{h^2} + \omega^2 u_{i,j} = f_{i,j} + s_{(2i+1,2j)} \quad .$$

- For $i \in (2, N-1)$ and $j = 1$ or $j = N$, using the Helmholtz equation and the Neumann boundary condition

- Case $j = 1$,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - u_{i,j}}{h^2} + \omega^2 u_{i,j} = f_{i,j} \quad .$$

- Case $j = N$,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j-1} - u_{i,j}}{h^2} + \omega^2 u_{i,j} = f_{i,j} \quad .$$

Note that we only solve the Helmholtz equation inside the domain Ω , and if some cell (i, j) which is inside Ω ($R(i, j) = -1$) and it has one of four neighbours outside, then we will use the boundary condition to approximate its neighbours.

We also use this scheme to solve the adjoint problem.

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